All groups considered are finite.

Let $G$ be a group, $p$ a prime dividing the order of $G$ and $P$ a Sylow $p$-subgroup of $G$. If there exists a normal subgroup $N$ of $G$ such that $G = PN$ and $P \cap N = 1$, then $G$ is called a $p$-nilpotent group, and in this case, $N$ is called a normal $p$-complement of $G$. It is well-known that in the group theory there are many mathematicians to investigate the existence of normal $p$-complements of groups, and therefore there are many results on the existence of normal $p$-complements, including some famous theorems, for example, Frobenius Theorem, Burnside Theorem, Thompson Theorem, Glauberman Theorem and so on. Now we specially point out the following earliest result about $p$-nilpotent groups due to Burnside.

**Burnside Theorem.** Let $G$ be a group and $P$ a Sylow $p$-subgroup of $G$. If $N_G(P) = C_G(P)$, then $G$ is $p$-nilpotent.

Clearly, the assumption “$N_G(P) = C_G(P)$” in Burnside theorem is very shorter. It is the reason that many mathematicians hope to generalize Burnside theorem. Since the assumption “$N_G(P) = C_G(P)$” is equivalent to the condition “$P \leq Z(N_G(P))$”, Sylow $p$-subgroups of $G$ must be abelian and every element of $P$ commutes with every element of $N_G(P)$. Hence the following result due to Hall is a
generalization.

Theorem $A^{[H]}$. Let $P$ be a Sylow $p$-subgroup of a group $G$. If every $p'$-element of $N_G(P)$ commutes with every element of $P$ and the nilpotent class of $P$ is less $p$, then $G$ is $p$-nilpotent.

Wielandt gave further generalization of Burnside theorem.

Theorem $B^{[W]}$. Let $P$ be a Sylow $p$-subgroup of a group $G$. If $P$ is regular, then the largest $p$-factor group of $N_G(P)$ is isomorphism to the largest $p$-factor group of $G$. Therefore, if $N_G(P)$ is $p$-nilpotent, then $G$ must be $p$-nilpotent.

At the same time, Burnside theorem tell us that the property of $N_G(P)$ can strongly influence the structure of finite groups. The following results further illustrate this.

Theorem $C^{[BMH]}$. A group $G$ is nilpotent if and only if $N_G(P)$ is nilpotent for every Sylow subgroup $P$ of $G$.

Ballester-Bolinches and Shemetkov further obtain a nice result in this line.

Theorem $D^{[BS]}$. A group $G$ is nilpotent if and only if $N_G(P)$ is $p$-nilpotent for every prime $p$ dividing the order of $G$ and every Sylow $p$-subgroup $P$ of $G$.

On the other hand, it is well-known that maximal subgroups play an important part in the group theory, so do minimal subgroups. Hence in recent years many authors investigate the influence of minimal subgroups on the structure of groups. If we read these papers, it is not difficult to know that many results are based on the following Itô’s lemma.

Itô’s Lemma. Let $p$ be a prime dividing the order of a group $G$ and every element of $G$ with order $p$ is contained in the center of $G$. 

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1) if \( p > 2 \), then \( G \) is \( p \)-nilpotent.

2) if \( p = 2 \) and every element of \( G \) with order 4 is also contained in the center of \( G \), then \( G \) is \( p \)-nilpotent.

The above facts tell us that Burnside theorem and Itô lemma are two basic results in the group theory and the property of \( N_G(P) \) can strongly influence the structure of groups for the Sylow \( p \)-subgroup \( P \) of \( G \). It is inspired by these two basic results that we hope to determine the structure of finite groups. Now I introduce our recently results based on the normality and complementarity of some minimal subgroups in \( N_G(P) \) for a Sylow subgroup \( P \) of a group \( G \).

1 The unity of Burnside theorem and Itô lemma

Now we recall Burnside theorem and Itô Lemma first. Burnside theorem states that

**If** \( P \leq Z(N_G(P)) \), **then** \( G \) **is** \( p \)-**nilpotent**.

Itô lemma states that

**\( G \) is \( p \)-**nilpotent if \( A \leq Z(G) \) for every minimal \( p \)-subgroup \( A \) and every cyclic subgroup \( A \) of order 4 if \( p = 2 \).**

So it is natural to ask whether \( G \) is \( p \)-nilpotent if \( A \leq Z(N_G(P)) \) for every minimal \( p \)-subgroup \( A \) and every cyclic subgroup \( A \) of order 4 if \( p = 2 \).

The answer to the above question is correct. In fact, Ballester-Bolinches and Guo have recently given an answer to this question. We proved the following result.

**Theorem 1.1**[\( BG1 \)]. Let \( p \) be a prime dividing the order of a group \( G \) and \( P \) a Sylow \( p \)-subgroup of \( G \). If every element of \( P \cap O^p(G) \) with order \( p \) lies in the center of \( N_G(P) \) and when \( p = 2 \) either every element of \( P \cap O^p(G) \) with order 4 also lies
in the center of $N_G(P)$ or $P$ is quaternion-free and $N_G(P)$ is 2-nilpotent, then $G$ is $p$-nilpotent.

Theorem 1.1 not only unity Burnside Theorem and Itô Lemma but also generalize these two basic Theorems. Now recall that a subgroup $H$ of a group $G$ is permutable (or quasinormal) in $G$ if $HK = KH$ for any subgroup $K$ of $G$. It is clear that permutability is a weak form of normality. Hence a natural question arises:

Is a group $G$ still $p$-nilpotent if every subgroup of $P \cap O^p(G)$ with order $p$ is permutable in $N_G(P)$ and when $p = 2$ every cyclic subgroup of $P \cap O^p(G)$ with order 4 is also permutable in $N_G(P)$?

Our recent work gives an answer to this question. First we have the following example which illustrates that the answer to the above question is negative.

**Example.** Let $G = A_5$, the alternating group of degree 5, and $P$ a Sylow 5-subgroup of $G$. Then it is clear that $P \cap O^p(G) = P$ and $N_G(P)$ is a subgroup of $G$ with order 10. Hence every minimal subgroup of $P$ is permutable in $N_G(P)$. But $G$ is a simple group.

However, we can have the following Theorem.

**Theorem** 1.2$^{[GS1]}$. Let $p$ be a prime number dividing the order of a group $G$ and $P$ a Sylow $p$-subgroup of $G$. Assume that every minimal subgroup of $P \cap O^p(G)$ is permutable in $P$ and $N_G(P)$ is $p$-nilpotent. Assume that in addition when $p = 2$, either $[\Omega_2(P \cap O^p(G)), P] \leq \Omega_1(P \cap O^p(G))$ or $P$ is quaternion-free. Then $G$ is $p$-nilpotent.

By using our Theorem 1.2, we may get a series of corollaries for $p$-nilpotence, some of which contain not only Itô’s lemma and Burnside’s theorem for $p$-nilpotence but also Ballester-Bolinches and Guo’s result$^{[BG1]}$. Hence Theorem 1.2 contain not only Itô’s lemma and Burnside’s theorem for $p$-nilpotence but also Ballester-
Bolinches and Guo’s result [BG1]. Now we list the following one.

**Theorem 1.3.** Let $p$ be a prime dividing the order of a group $G$ and $P$ a Sylow $p$-subgroup of $G$. If $\Omega_1(P \cap O^p(G))$ is contained in the center of $N_G(P)$ and when $p = 2$ either $[\Omega_2(P \cap O^p(G)), N_G(P)] \leq \Omega_1(P \cap O^p(G))$ or $P$ is quaternion-free, then $G$ is $p$-nilpotent.

Now, we may give some applications of Theorem 1.2. A well known result of Thompson asserts that a group $G$ is solvable if $G$ has a nilpotent maximal subgroup of odd order. Later on, there are many various generalizations of this result. We mention the following one. K. Brown [B] in 1971 proved that if a solvable group $A$ acts on a group $G$ which has a nilpotent maximal $A$-invariant subgroup $M$ with an abelian Sylow 2-subgroup, then $G$ is solvable. Now we can prove the following theorem.

**Theorem 1.4.** Assume that a solvable group $A$ acts on a group $G$. If $G$ has a nilpotent maximal $A$-invariant subgroup $M$ which has a Sylow 2-subgroup $P$ with the property that every minimal subgroup of $\Omega_1(P \cap O^p(G))$ is permutable in $P$ and $[\Omega_2(P \cap O^p(G)), P] \leq \Omega_1(P \cap O^p(G))$, then $G$ is solvable.

If we remove the hypothesis that $A$ is solvable in Theorem 1.4, we can prove the following

**Theorem 1.5.** Let $G$ be a group on which a group $A$ acts. Assume that $M$ is a nilpotent maximal $A$-invariant subgroup of $G$ and $P$ is a Sylow 2-subgroup of $M$. If every minimal subgroup of $\Omega_1(P \cap O^p(G))$ is permutable in $P$, $[\Omega_2(P \cap O^p(G)), P] \leq \Omega_1(P \cap O^p(G))$ and one of the following conditions is satisfied, then $G$ is solvable.

(i) There is an $A$-invariant Sylow $q$-subgroup $Q(\neq 1)$ of $G$ for some prime $q \in \pi(G) - \pi(M)$.

(ii) $(|A|, |G: M|) = 1$.

Finally, we will give another application of Theorem 1.2. We mention a result
of Buckley [BU]. He showed that a group of odd order is supersolvable if each minimal subgroup of $G$ is normal in $G$. Along these same lines, several authors have investigated the influence normality and permutability of the minimal subgroups on the structure of $G$. Now we prove a result which is a generalization of known results. First we remove the assumption for all minimal subgroups. We want to use few minimal subgroups to determine the structure of the group. Second we do not assume that minimal subgroups have some properties in group $G$. We only assume that minimal subgroups have some properties in a subgroup of $G$. In fact, we can prove the following results.

**Theorem 1.6.** Let $N$ be a normal subgroup of a group $G$ such that $G/N$ is supersolvable. Also let $p$ be any prime number dividing the order of $N$ and $P$ a Sylow $p$-subgroup of $N$. Assume that every minimal subgroup of $\Omega_1(P \cap O^p(G))$ is contained in the center of $P$ and when $p = 2$, assume in addition that $[\Omega_2(P \cap O^p(G)), P] \leq \Omega_1(P \cap O^p(G))$. Also assume that every minimal subgroup of $\Omega_1(P \cap O^p(G))$ is $S$-quasinormal in $N_G(P)$ and when $p = 2$, every cyclic subgroup of order 4 of $\Omega_2(P \cap O^p(G))$ is $S$-quasinormal in $N_G(P)$. Then $G$ is supersolvable.

We can also generalize Theorem 1.6 to a saturated formation containing the class of all supersolvable groups.

### 2 Complementarity of minimal subgroups

Let $G$ be a group. A subgroup $H$ of $G$ is said to be complemented in $G$ if there exists a subgroup $K$ of $G$ such that $G = HK$ and $H \cap K = 1$. It is quite clear that the existence of complements for some families of subgroups of a group give a lot of information about its structure. For instance, P. Hall in 1937 proved the following result.

**Theorem $E^{[H^3]}$**. A group $G$ is solvable if and only if every Sylow subgroup of $G$ is complemented.
This result is very important so that it is regard as a fundamental work of solvable groups, and several generalizations followed. Now we specially point out the following result proved by Arad and Ward.

**Theorem** $F^{[AW]}$. A group $G$ is solvable if and only if every Sylow 2-subgroup and every Sylow 3-subgroup of $G$ are complemented.

In 1937 P. Hall in [H2] ever investigated the groups in which every subgroup is complemented. He proved that this kind of groups is supersolvable with elementary abelian Sylow subgroups. It is interesting that in 1997 Ballester-Bolinches and Guo [BG2] found that every subgroup of a group is complemented if and only if every minimal subgroup is complemented. That means the following result is true.

**Theorem** $G^{[BG2]}$. A group $G$ is supersolvable with elementary abelian Sylow subgroups if and only if every minimal subgroup of $G$ is complemented.

This fact illustrates that the complementarity of minimal subgroups have strong influence on the structure of a group. We are lead to investigate the structure of groups by means of the complementarity of minimal subgroups. Our idea is to use few minimal subgroups to determine the structure of groups. In fact, we drop the assumption that every minimal subgroup is complemented, but instead, we only use fewer complemented minimal subgroups to determine the structure of the group. On the other hand, we drop the assumption that the minimal subgroups must be complemented in the whole group $G$, but instead, we only assume that the minimal subgroups are complemented in a subgroup of $G$. Our first result is the following result about $p$-nilpotency.

**Theorem** 2.1$^{[GS2]}$. Let $G$ be a group. Let $p$ be a prime dividing the order of $G$ and $P$ a Sylow $p$-subgroup of $G$. If every minimal subgroup of $P \cap O^p(G)$ is complemented in $N_G(P)$ and $N_G(P)$ is $p$-nilpotent, then $G$ is $p$-nilpotent.

For this result, we have the two remarks:
**Remark 1.** The assumption that $N_G(P)$ is $p$-nilpotent in above theorem cannot be removed. In fact, if we take $G = A_5$, the alternating group of degree 5, then it is easy to see that $N_G(P)$ is a subgroup of $G$ of order 10 for every Sylow 5-subgroup $P$ of $G$. Hence every minimal subgroup of order 5 in $P$ has a complement in $N_G(P)$. However, $G = A_5$ is simple.

**Remark 2.** If we assume that $p$ is the smallest prime dividing the order of $G$, the assumption that $N_G(P)$ is $p$-nilpotent in above theorem can be removed. In fact, if $p$ is the smallest prime dividing the order of $G$ and every minimal subgroup of $P \cap O^p(G)$ is complemented in $N_G(P)$, then $G$ is $p$-nilpotent.

Using Theorem 2.1 and Remark 2, we give some applications, which have their independent interests. For example,

**Theorem 2.2.** Let $N$ be a normal subgroup of a group $G$ and $p$ a prime dividing the order of $N$ and $P$ a Sylow $p$-subgroup of $N$. Also let $\mathcal{F}$ be a saturated formation containing the class $\mathcal{N}_p$ of all $p$-nilpotent groups and $G/N \in \mathcal{F}$. If $N_G(P)$ is $p$-nilpotent and every minimal subgroup of $P \cap O^p(G)$ is complemented in $N_G(P)$, then $G \in \mathcal{F}$.

**Theorem 2.3.** Let $\mathcal{F}$ be a formation containing $\mathcal{U}$, the class of supersolvable groups. Let $H$ be a normal subgroup of a group $G$ such that $G/H \in \mathcal{F}$. If for every Sylow subgroup $P$ of $H$, every minimal subgroup of $P \cap G^N$ is complemented in $N_G(P)$, then $G$ is in $\mathcal{F}$, where $G^N$ is the nilpotent residual of $G$.

If $G$ is assumed to be a solvable group, then the number of complemented minimal subgroups in Theorem 2.3 can be further reduced. In fact, we have the following theorem.

**Theorem 2.4.** Let $\mathcal{F}$ be a formation containing $\mathcal{U}$, the class of supersolvable groups. Let $H$ be a normal subgroup of a solvable group $G$ such that $G/H \in \mathcal{F}$. 
If every minimal subgroup of the Fitting subgroup $F(G^N \cap H)$ of $G^N \cap H$ has a complement in $G$, then $G$ belongs to $\mathcal{F}$, where $G^N$ is the nilpotent residual of $G$.

**Remark 3.** We should point out that in Theorem 2.3 and 2.4 we only assume that $\mathcal{F}$ is a formation, which is different from the other cases in which we have to assume that $\mathcal{F}$ is a saturated formation.

**Remark 4.** If we take $\mathcal{F}$ and $H$ special, then we get some interesting results. For example, if we let $\mathcal{F}$ be the class of supersolvable groups and $H = G^N$, then, by Theorem 2.3 and Theorem 2.4, some sufficient conditions for a group to be supersolvable are obtained.

We also investigate the groups in which every maximal subgroup is supersolvable group with elementary abelian Sylow subgroups, we call this class of groups BNS. First we have the following results.

**Theorem 2.5.** Let $G$ be a group. If every maximal subgroup of $G$ is in $\mathcal{BNS}$ and $|\pi(G)| \geq 4$, then $G$ is in $\mathcal{BNS}$.

**Theorem 2.6.** Let $G$ be a group. If for every Sylow subgroup $P$ of $G$, $N_G(P)$ is a $\mathcal{BNS}$-group, then $G$ is a $\mathcal{BNS}$-group.

Then, we give a complete classification of this kind groups, that is

**Theorem 2.7.** Let $G$ be a group. If every maximal subgroup of $G$ belongs to $\mathcal{BNS}$ but $G$ does not, then one of the following statements is true.

(I) $G = \langle a | a^{p^2} = 1 \rangle$, where $p$ is a prime.

(II) $G = \langle a, b \rangle$ with $a^p = b^p = c^p = 1, [a, b] = c, [a, c] = [b, c] = 1$, where $P$ is an odd prime.

(III) $G = P \ltimes Q$ is a minimal non-abelian group with order $pq^\alpha$ and $p \not| (q - 1)$, where $P \in \text{Syl}_p G, Q \in \text{Syl}_q G$, $p$ and $q$ are distinct primes and $\alpha > 1$ is a natural
number.

(IV) $G$ is a minimal non-supersolvable group with order $pqr^p$, $pq|(r - 1)$ and $p|(q - 1)$, where $G$ equals $\langle c_1, c_2, ..., c_p, a, b \rangle$ satisfying the following relations

$$c_1^r = c_2^r = ... = c_p^r = 1, c_i c_j = c_j c_i (i, j = 1, 2, ..., p), \quad a^p = b^q = 1, b^a = b^u;$$

the order of $u(\text{mod } q)$ is $p$, $c_i^a = c_{i+1}, (i = 1, 2, ..., p - 1)$, $c_p^a = c_1, c_i^b = c_i^{p^{p-i+1}}, i = 1, ..., p$; the order of $v(\text{mod } r)$ is $q$, and $p$, $q$ and $r$ are distinct primes.

Conversely, every group $G$ of type (I), (II), (III) and (IV) does not belong to $\mathcal{BNS}$ but every maximal subgroup of $G$ belongs to $\mathcal{BNS}$.

We also investigate the group $G$ in which there exists a maximal subgroup $M$ such that every minimal subgroup of $M$ is complemented in $G$. In fact we show the following result:

**Theorem 2.8.** Let $G$ be a group and $M$ a maximal subgroup of $G$. If every minimal subgroup of $M$ has a complement in $G$, then $G$ is solvable.

This result is somehow analogous to the well-known Thompson theorem. Furthermore, we investigate the groups which has two maximal subgroups with the above properties and therefore some conditions for a group to be supersolvable are given. For example, we have the following results.

**Theorem 2.9.** Let $M_1$ and $M_2$ be two maximal subgroups of a group $G$. If $M_1$ and $M_2$ are not conjugates in $G$ and every minimal subgroup of $M_i$ ($i = 1, 2$) has a complement in $G$, then $G$ is supersolvable.

**Theorem 2.10.** Let $M_1$ and $M_2$ be maximal subgroups of a group $G$. If $[G : M_1] \neq [G : M_2]$ and every minimal subgroup of $M_i$ ($i = 1, 2$) has a complement in $G$, then $G$ is supersolvable.
References


