Categories of Contexts

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Abstract

We provide the categorical framework for some fundamental tools and facts in Formal Concept Analysis, the theory of contexts and their concept lattices. Morphisms between contexts are certain pairs of maps, one between the objects and one between the attributes of the contexts in question. We study several natural classes of morphisms and the connections between them. Among other things, we show that the category \( \text{Clc} \) of complete lattices with complete homomorphisms is (up to a natural isomorphism) fully reflective in the category of contexts with so-called conceptual morphisms; the reflector associates with each context its concept lattice. On the other hand, we find a dual adjunction between \( \text{Clc} \) and the category of contexts with so-called concept continuous morphisms. Suitable restrictions of the adjoint functors yield categorical equivalences and dualities between purified contexts and doubly based lattices, and in particular between reduced contexts and small-based complete lattices. A central role is played by continuous maps between closure spaces and by adjoint maps between complete lattices.

Mathematics Subject Classification:

Key words: Adjunction, complete lattice, complete homomorphism, context, concept, conceptual morphism, continuous.

0 Introduction

Fundamental in Formal Concept Analysis (FCA) is the interplay between so-called contexts (i.e. triples \((J, M, I)\) constituted by a set \(J\) of ‘subjects’ or ‘objects’, a set \(M\) of ‘marks’ or ‘attributes’, and an incidence relation \(I\) between them, interpreting \(jIm\) as ‘\(j\) has the attribute \(m\)’, or ‘\(m\) holds for \(j\)’) and the associated concept lattices introduced by Wille [14]. It is therefore of primary interest to elucidate the passage between contexts and concept lattices – or, categorically speaking, to investigate the relevant functors between the involved categories. Natural candidates for morphisms on
the lattice side are maps that preserve arbitrary joins, or meets, or both. Usually, morphisms between contexts are \textit{pairs} of maps, because contexts have two ground sets. Since either of them carries a natural closure system (that of ‘extents’, isomorphic to the concept lattice, and that of ‘intents’, dual to the former), it is rather obvious that \textit{continuity} will play a crucial role in that setting (see [5] for categorical correspondences between closure spaces and complete lattices). Continuity is also the defining condition for ‘scalings’ in measurement theory (see [10, 14]).

Here, we are mainly interested in \textit{complete homomorphisms} between concept lattices. Since they preserve both joins and meets, we have to take pairs of continuous maps between the underlying contexts – but that is not enough: one needs certain links between the two involved mappings. This leads us to two essentially different but equally important notions, that of \textit{conceptual morphisms} and that of \textit{concept continuous} morphisms. The category \textbf{Clc} of complete lattices with complete homomorphisms turns out to be a full reflective subcategory of the category of contexts with conceptual morphisms – just by passing from contexts to their concept lattices. But there is also a \textit{dual} adjunction between the category \textbf{Clc} and the category of contexts with concept continuous morphisms. Various results on subcontexts and their concept lattices are immediate consequences. Modifying the adjoint functors, we shall arrive at a categorical equivalence and a duality between purified contexts and so-called doubly based lattices – in particular, between reduced contexts and irreducibly bigenerated complete lattices.

For applications, it is helpful that the relevant morphisms may be described in \textit{fist order terms}: given contexts \((J,M,I)\) and \((H,N,R)\), a pair of mappings \(\alpha: J \rightarrow H\) and \(\beta: M \rightarrow N\) is \textit{conceptual} iff it preserves incidence and an object \(h\) has the attribute \(n\) whenever each object having all attributes whose \(\beta\)-image holds for \(h\) also has the attributes of each object whose \(\alpha\)-image has the attribute \(n\). On the other hand, \((\alpha, \beta)\) is \textit{concept continuous} iff it reflects incidence, an attribute \(n\) holds for \(\alpha(j)\) whenever each attribute whose \(\beta\)-image generalizes \(n\) holds for \(j\), and \(\beta(m)\) holds for an object \(h\) whenever \(m\) holds for each object whose \(\alpha\)-image specializes \(h\) (where an attribute \textit{generalizes} another one if it holds for all objects having that attribute, and an object specializes another one if it shares all of its attributes). For a succinct formalization of these somewhat longwinded verbal definitions, see Propositions 3.1 and 3.2.

Parts of the subsequent material have already been established and presented at conferences in the eighties, or incorporated in other papers [2, 6, 7, 8, 9] and in the monograph by Ganter and Wille [10], which is recommended as the universal source for the mathematical treatment of FCA. The recent vital correspondence with workers in the field of FCA persuaded me to elaborate the categorical background systematically, and here is the output.
1 Categories of Complete Lattices

In this preliminary section, we summarize some definitions and known facts about ordered sets, complete lattices and morphisms between them.

Given an arbitrary map $\varphi : X \rightarrow Y$, we shall denote by $\varphi[A]$ the image of $A$ and by $\varphi^{-1}[B]$ the preimage of $B \subseteq Y$ under $\varphi$.

Recall that a map $\varphi : P \rightarrow Q$ between (partially) ordered sets is
- order preserving or isotone if $x \leq y \Rightarrow \varphi(x) \leq \varphi(y)$,
- order reflecting or antitone if $x \leq y \Leftrightarrow \varphi(x) \leq \varphi(y)$,
- an order embedding if $x \leq y \Rightarrow \varphi(x) \leq \varphi(y)$.

Furthermore, $\varphi$ is join-dense if each element in the codomain $Q$ is a join (supremum, least upper bound) of elements in the range of $\varphi$, or equivalently, if for $q \nleq r \in Q$ there is a $p \in P$ with $\varphi(p) \leq q$ but $\varphi(p) \nleq r$. Caution: the composition of two join-dense (isotone) maps need not be join-dense! Meets and meet-dense maps are defined dually.

Of particular importance for our considerations are adjoint maps and functors (see e.g. [3] or [11] for the order-theoretical and [1] for the categorical background). A pair of maps $\varphi : P \rightarrow Q$ and $\psi : Q \rightarrow P$ between ordered sets is adjoint if

$$\varphi(p) \leq q \iff p \leq \psi(q)$$

for all $p \in P$ and $q \in Q$. In that situation, $\varphi$ is the left or lower adjoint of $\psi$, which in turn is the right or upper adjoint of $\varphi$. By antisymmetry of the order relations, lower and upper adjoints determine each other uniquely; we write $\varphi^*$ for the upper adjoint of $\varphi$, and $\psi_*$ for the lower adjoint of $\psi$ (some authors use the converse notation). Lower adjoints preserve arbitrary joins, and upper adjoints arbitrary meets. It is helpful to know that a lower adjoint is injective iff its upper adjoint is surjective, and vice versa. An injective lower adjoint $\varphi$ (upper adjoint $\psi$) is always an order embedding and satisfies $\varphi^* \circ \varphi = id$ ($\psi_* \circ \psi = id$). Note also that any join-dense join-preserving map is already surjective. Moreover, a map between posets is an isomorphism iff it has both an upper and a lower adjoint and these two adjoints coincide.

Given subsets $A, B$ of a poset $P$, we denote by $A^\uparrow$ the collection of all upper bounds of $A$ and by $B^\downarrow$ that of all lower bounds of $B$. In particular,

$$\downarrow x = \{x\}^\downarrow = \{p \in P : p \leq x\} \quad \text{and} \quad \uparrow x = \{x\}^\uparrow = \{p \in P : p \geq x\}$$

are the principal ideal and the principal filter generated by $x \in P$, respectively. More generally, for $A, B \subseteq P$,

$$A^\downarrow = \bigcap \{\downarrow x : A \subseteq \downarrow x\}$$

is the lower cut generated by $A$, and

$$B^\uparrow = \bigcap \{\uparrow x : B \subseteq \uparrow x\}$$
is the upper cut generated by $B$. The cuts in the sense of MacNeille [12] (generalizing Dedekind’s cuts of rational numbers) are the pairs $(A, B)$ with $A = B^\downarrow$ and $B = A^\uparrow$. Ordered by $(A, B) \leq (C, D) \iff A \subseteq B \iff D \subseteq C$, they form a complete lattice, the famous Dedekind-MacNeille completion, which is isomorphic to the closure system of lower cuts and dually isomorphic to the closure system of upper cuts (cf. [3, 4, 6, 12]).

A map between posets is a lower adjoint iff it is residuated, i.e. preimages of principal ideals are principal ideals, and it is an upper adjoint iff it is residual, i.e. preimages of principal filters are principal filters. Similarly, a map between posets is called lower (upper) cut continuous if preimages of lower (upper) cuts are again lower (upper) cuts. From [6], we cite:

**Theorem 1.1** Generally, one has the following implications:

- residuated $\Rightarrow$ lower cut continuous $\Rightarrow$ join preserving
- residual $\Rightarrow$ upper cut continuous $\Rightarrow$ meet preserving

and for maps between complete lattices, the converse implications hold, too. The completion by cuts yields a reflector from the category of posets with lower (upper) cut continuous maps to the full subcategory of complete lattices with join (meet) preserving maps.

We denote by $\mathbf{Clc}$ the category of complete lattices and complete homomorphisms, i.e. maps preserving arbitrary joins and meets. On the other hand, we have the category $\mathbf{Clc}_*$ of complete lattices and doubly residuated maps, i.e. maps $\varphi$ possessing an upper adjoint $\varphi^*$ which again has an upper adjoint $\varphi^{**}$, and the category $\mathbf{Clc}^*$ of complete lattices and doubly residual maps $\psi$, having a lower adjoint $\psi_*$ which again has a lower adjoint $\psi^{**}$. Passing to upper adjoints, one obtains a dual isomorphism between the categories $\mathbf{Clc}_*$ and $\mathbf{Clc}$, but also one between $\mathbf{Clc}$ and $\mathbf{Clc}^*$. Composing both duality functors, one arrives at an isomorphism between the categories $\mathbf{Clc}_*$ and $\mathbf{Clc}^*$, sending any doubly residuated map $\varphi$ to $\varphi^{**}$, and in the opposite direction, any doubly residual map $\psi$ to $\psi^{**}$. Note also that $\psi$ is an isomorphism iff it is doubly residuated (residual) with $\psi = \psi^{**}$ ($\psi = \psi_*$). Summarizing the previous remarks, we see that under the above functors, the following pairs of categories of complete lattices are duals of each other:

<table>
<thead>
<tr>
<th>category</th>
<th>morphisms</th>
<th>dual</th>
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<tbody>
<tr>
<td>$\mathbf{Clc}$</td>
<td>complete homomorphisms</td>
<td>$\mathbf{Clc}_*$</td>
<td>doubly residuated maps</td>
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<td></td>
<td></td>
<td>$\mathbf{Clc}^*$</td>
<td>doubly residual maps</td>
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<tr>
<td>$\mathbf{Cld}$</td>
<td>dense (= surjective) complete homomorphisms</td>
<td>$\mathbf{Cld}_*$</td>
<td>doubly residuated embeddings</td>
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<td></td>
<td>$\mathbf{Cld}^*$</td>
<td>doubly residual embeddings</td>
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<tr>
<td>$\mathbf{Cle}$</td>
<td>embedding (= injective) complete homomorphisms</td>
<td>$\mathbf{Cle}_*$</td>
<td>doubly residuated surjections</td>
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<td></td>
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<td>$\mathbf{Cle}^*$</td>
<td>doubly residual surjections</td>
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<td>isomorphisms</td>
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2 Closure Spaces and Continuous Maps

Since contexts and their concept lattices are always intimately related with certain closure structures, a few preliminary remarks about closure spaces and their morphisms are in order before starting the morphism theory for contexts and concept lattices. For more background concerning the interaction between closure spaces and complete lattices, we refer to [5].

A closure space is a set $X$ together with a closure system $\mathcal{C}$, that is, a collection of subsets that is closed under arbitrary intersections; thus, $X = \bigcap \emptyset \in \mathcal{C}$. For each subset $A$ of $X$, there is a least member of $\mathcal{C}$ containing $A$, denoted by $\overline{A}$ and called the closure of $A$. Clearly, $\mathcal{C}$ is a complete lattice in which arbitrary meets coincide with intersections (but joins not always with unions). There is a canonical map from $X$ to $\mathcal{C}$, $\eta_X : X \to \mathcal{C}, x \mapsto \overline{\{x\}}$.

Let us recall several equivalent definitions of continuity (see e.g. [5]):

**Proposition 2.1** For a map $\alpha$ between closure spaces $(X, \mathcal{C})$ and $(Y, \mathcal{D})$, the following conditions are equivalent:

(a) Preimages of closed sets under $\alpha$ are closed.
(b) $\alpha[\overline{A}] \subseteq \overline{\alpha[A]}$ for all $A \subseteq X$.
(c) There are adjoint maps $\alpha^- : \mathcal{C} \to \mathcal{D}$, $\alpha^+ : \mathcal{D} \to \mathcal{C}$ with $\alpha^- \circ \eta_X = \eta_Y \circ \alpha$.
(d) There is a join-preserving map $\alpha^- : \mathcal{C} \to \mathcal{D}$ with $\alpha^-(\overline{\{x\}}) = \overline{\{\alpha(x)\}}$.

Moreover, these maps $\alpha^-$ and $\alpha^+$ are uniquely determined:

$\alpha^-(C) = \overline{\alpha[C]}$, $\alpha^+(D) = \overline{\alpha[D]}$.

In order to determine under what conditions the maps $\alpha^-$ and $\alpha^+$ are injective or surjective, respectively, we say $\alpha$ is

- (strictly) dense if for each $B \subseteq Y$ there is some $A \subseteq X$ with $B = \overline{\alpha[A]}$,
- full if $\alpha^-[\overline{\alpha[A]}] \subseteq A$ for all $A \subseteq X$, i.e. $\alpha(x) \in \overline{\alpha[A]}$ implies $x \in A$.

We shall omit the word ‘strictly’. The following facts are easily checked:

**Lemma 2.1** A continuous map $\alpha$ between closure spaces $X$ and $Y$ is

- dense iff $\alpha^-$ is surjective iff $\alpha^+$ is injective,
- full iff $\alpha^-$ is injective iff $\alpha^+$ is surjective.

Furthermore, $\alpha$ is full and continuous iff it is initial, i.e. $\mathcal{C} = \alpha^-[\mathcal{A}(Y)]$.

The term “full” is common in measurement theory (see e.g. [10]), but in view of the last lemma, “faithful” perhaps would fit better with the categorical meaning.
3 Morphisms Between Contexts

A (formal) context is a triple $K = (J, M, I)$ where $I$ is a relation between elements of $J$ and elements of $M$, i.e. $I \subseteq J \times M$. For $A \subseteq J$ and $B \subseteq M$, put

$$A^\uparrow = \{ m \in M : jIm \text{ for all } j \in A \},$$
$$B^\downarrow = \{ j \in J : jIm \text{ for all } m \in B \}.$$

Instead of $\{j\}^\uparrow$ and $\{m\}^\downarrow$, we shall write $j^\uparrow$ and $m^\downarrow$, respectively. The complementary relation $J \times M \setminus I$ will be denoted by $\neg I$. A (formal) concept of the context $K$ is a pair $(A, B)$ with $A \subseteq J$, $B \subseteq M$, $A = B^\downarrow$ (the ‘extent’) and $A^\uparrow = B$ (the ‘intent’). Ordered by

$$(A, B) \leq (C, D) \iff A \subseteq C \iff D \subseteq B,$$

the concepts form a complete lattice, the so-called concept lattice $B^K$. By passing to the first or second components, this lattice is isomorphic to the closure system $E^K$ of all extents and dually isomorphic to the closure system $I^K$ of all intents. Thus, concept lattices are the natural generalization of Dedekind-MacNeille completions, replacing orders by arbitrary relations. Again, the corresponding closure operators are $^\uparrow$ and $^\downarrow$, respectively. One writes $A \rightarrow B$ for $A^\uparrow \subseteq B^\downarrow$, i.e. $B \subseteq A^\downarrow$, meaning that the objects in $B$ share all common properties of objects in $A$.

Naturally, context morphisms have to be certain pairs of maps, one between the objects and the other between the attributes. The choice of morphisms is not evident and may depend heavily on the intended investigations and results. From the Galois-theoretical point of view, it is reasonable to consider pairs of maps with opposite directions (and that has been done by several authors). However, we shall not pursue that trace in the present considerations but focus on situations where both maps run into the same direction - an approach that leads to quite satisfactory results as well.

Given two contexts $K = (J, M, I)$ and $L = (H, N, R)$, a pair $(\alpha, \beta)$ of maps $\alpha : J \rightarrow H$ and $\beta : M \rightarrow N$ will be referred to as a mapping pair or (weak) concept morphism. In accordance with the corresponding general closure-theoretical definitions, we say $\alpha$ is

- (extent) continuous if preimages of extents under $\alpha$ are extents,
- (extent) dense if for all $C \subseteq H$, there is an $A \subseteq J$ with $C^{\uparrow} = \alpha[A]^{\uparrow}$,
- (extent) full if for all $A \subseteq J$, $\alpha(j) \in \alpha[A]^{\uparrow}$ implies $j \in A^{\uparrow}$.

Dually defined are (intent) continuous, dense and full maps, respectively. Extent continuous maps are often interpreted as scalings in the theory of measurement, in particular if the elements of the codomain are numbers or numerical functions (see e.g. [10] and [14]).

Although every closure space $(X, A)$ may be regarded as an extent space, namely of the context $(X, A, \in)$, there is a crucial difference between arbi-
trary closure spaces and extent or intent spaces: in the latter situation the various types of morphisms admit descriptions in first order terms, involving quantification over objects and attributes only, but not over subsets (like extents or intents). This reduction of complexity is one of the major advantages of Formal Concept Analysis. Note that statements like \( h^\uparrow \subseteq \alpha(j)^\uparrow \) or \( \alpha^{-}[n^\downarrow] \subseteq m^\downarrow \) are expressible in first order terms (the former meaning that \( hRn \) implies \( \alpha(j)Rn \), and the latter that \( \alpha(j)Rn \) implies \( jIm \)).

In the sequel, \( (\alpha,\beta) \) always denotes a mapping pair between contexts \( \mathcal{K} = (J,M,I) \) and \( \mathcal{L} = (H,N,R) \). The elementary proofs are omitted.

**Lemma 3.1** The following are equivalent:

(a) \( \alpha \) is extent continuous.
(b) \( \alpha^{-}[n^\downarrow] \) is an extent for each \( n \in N \).
(c) \( \alpha(j)Rn \) implies \( jIm \) for some \( m \in M \) with \( \alpha^{-}[n^\downarrow] \subseteq m^\downarrow \).
(d) \( \alpha[A]^\uparrow = \alpha[A]^\downarrow \) for all \( A \subseteq J \).
(e) \( \alpha[A]^\uparrow \subseteq \alpha[A]^\downarrow \) for all \( A \subseteq J \).
(f) \( A \rightarrow B \) implies \( \alpha[A] \rightarrow \alpha[B] \).

Dual characterizations hold for intent continuous maps.

**Lemma 3.2** The map \( \alpha \) is extent dense

iff for each \( h \in H \) there is a set \( A \subseteq J \) with \( h^\uparrow = \alpha[A]^\uparrow \)

iff \( hRn \) implies \( \alpha(j)Rn \) for some \( j \in J \) with \( h^\uparrow \subseteq \alpha(j)^\uparrow \).

Dual characterizations hold for intent density.

Consequently, the pair \( (\alpha,\beta) \) is dense (i.e. both \( \alpha \) and \( \beta \) are dense)

iff for \( hRn \) there exist \( j,m \) with \( \alpha(j)R\beta(m) \), \( h^\uparrow \subseteq \alpha(j)^\uparrow \), \( n^\downarrow \subseteq \beta(m)^\downarrow \).

Note also that the map \( \alpha \) is extent dense iff the set \( \{\alpha(j)^\downarrow : j \in J\} \) is join-dense in the extent lattice \( \mathcal{E}_\mathcal{L} \), and dually, \( \beta \) is intent dense iff the set \( \{\beta(m)^\uparrow : m \in M\} \) is meet-dense in the intent lattice \( \mathcal{I}_\mathcal{L} \).

**Lemma 3.3** The map \( \alpha \) is extent full

iff \( jIm \) implies \( \alpha(j)Rn \) for some \( n \in N \) with \( \alpha[m^\downarrow] \subseteq n^\downarrow \)

iff \( \alpha[A] \rightarrow \alpha[B] \) entails \( A \rightarrow B \)

iff each extent of \( \mathcal{K} \) is the preimage of an extent of \( \mathcal{L} \) under \( \alpha \).

Dual characterizations hold for intent fullness.

**Corollary 3.1** \( \alpha \) is initial, i.e. extent continuous and full

iff \( A \rightarrow B \) is equivalent to \( \alpha[A] \rightarrow \alpha[B] \)

iff the extents of \( \mathcal{K} \) are precisely the preimages of extents of \( \mathcal{L} \).
We come now to the crucial definitions, relating both partners of a mapping pair to each other. The mapping pair \((\alpha, \beta)\) is called

- **incidence preserving** if \(jIm\) implies \(\alpha(j)R\beta(m)\),
- **incidence reflecting** if \(\alpha(j)R\beta(m)\) implies \(jIm\),
- a **quasi-embedding** if it preserves and reflects incidence.
- an **embedding** if it is a quasi-embedding and both maps are injective.

**Lemma 3.4** The mapping pair \((\alpha, \beta)\) preserves incidence
\[
\begin{align*}
\text{iff } & \alpha[A]^{\uparrow} \subseteq \beta[A]^1 \text{ for all } A \subseteq J \\
\text{iff } & \beta[B]^{\uparrow} \subseteq \alpha[B]^1 \text{ for all } B \subseteq M.
\end{align*}
\]

**Lemma 3.5** The mapping pair \((\alpha, \beta)\) reflects incidence
\[
\begin{align*}
\text{iff } & \alpha^{-}[C^{\uparrow}] \subseteq \beta^{-}[C^1] \text{ for all } C \subseteq H \\
\text{iff } & \beta^{-}[D^{\uparrow}] \subseteq \alpha^{-}[D^1] \text{ for all } D \subseteq N.
\end{align*}
\]

Even more important than the above properties of mapping pairs are certain strong kinds of continuity. We say a mapping pair \((\alpha, \beta)\) is

- **(separately) continuous** if both \(\alpha\) and \(\beta\) are continuous,
- **concept preserving** if \((A, B) \in \mathcal{B}K\) implies \((\beta[B]^1, \alpha[A]^1) \in \mathcal{B}L\),
- **conceptual** if it is separately continuous and concept preserving,
- **concept continuous** if \((C, D) \in \mathcal{B}L\) implies \((\alpha^{-}[C], \beta^{-}[D]) \in \mathcal{B}K\),
- a **context isomorphism** if it is an embedding and \(\alpha, \beta\) are bijective.

**Proposition 3.1** The mapping pair \((\alpha, \beta)\) is conceptual
\[
\begin{align*}
\text{iff } & \alpha[A]^{\uparrow} = \beta[A]^1 \text{ for all } A \subseteq J \text{ and } \beta[B]^{\uparrow} = \alpha[B]^1 \text{ for all } B \subseteq M \\
\text{iff } & (\alpha, \beta) \text{ preserves incidence, } \beta[\alpha^{-}[n^1]]^1 \subseteq n^1 \text{ and } \alpha[\beta^{-}[h^1]]^1 \subseteq h^1 \\
\text{iff } & (\alpha, \beta) \text{ preserves incidence and for } h \subseteq n, \text{ there are } j \in J, m \in M \text{ with } jIm, \alpha^{-}[n^1] \subseteq m^1 \text{ and } \beta^{-}[h^1] \subseteq j^1.
\end{align*}
\]

**Proof.** Suppose \((\alpha, \beta)\) is conceptual. For \(A \subseteq J\), the pair \((A^{\uparrow}, A^1)\) is a concept; hence \((\beta[A]^1, \alpha[A]^1)\) is a concept, too. Thus \(\beta[A]^1 = \alpha[A]^{\uparrow} = \alpha[A]^1\) (by continuity of \(\alpha\)). The second equation is obtained analogously.

Next, if we assume the equations \(\alpha[A]^{\uparrow} = \beta[A]^1\) and \(\beta[B]^{\uparrow} = \alpha[B]^1\) then by Lemma 3.4, \((\alpha, \beta)\) preserves incidence; furthermore, we have \(\beta[\alpha^{-}[n^1]]^1 = \alpha[\alpha^{-}[n^1]]^1 \subseteq n^{\uparrow} = n^1\) and \(\alpha[\beta^{-}[h^1]]^1 = \beta[\beta^{-}[h^1]]^1 \subseteq h^1\).

On the other hand, if \(\beta[\alpha^{-}[n^1]]^1\) is contained in \(n^1\) then \(h \subseteq \beta[\alpha^{-}[n^1]]^1\), so we find an \(m \in \alpha^{-}[n^1]\) with \(\beta(m) \notin h^1\); it follows that \(\alpha^{-}[n^1] \subseteq m^1\), and if \(\alpha[\beta^{-}[h^1]]^1 \subseteq h^1\) then \(\beta(m) \notin \alpha[\beta^{-}[h^1]]^1\); therefore, we find a \(j \in \beta^{-}[h^1]\) with \(\alpha(j)R\beta(m)\), hence \(jIm\) (by incidence preservation).

Finally, let us suppose that \((\alpha, \beta)\) preserves incidence and for \(h \subseteq n\), there are \(jIm\) with \(\alpha^{-}[n^1] \subseteq m^1\) and \(\beta^{-}[h^1] \subseteq j^1\). Again by Lemma 3.4, we have
$\alpha[A]^{1} \subseteq \beta[A]^{1}$ and $\beta[B]^{1} \subseteq \alpha[B]^{1}$. Assume $h \notin \alpha[A]^{1}$; then there is some $n \in \alpha[A]^{1}$ with $hRn$. Choose $jI m$ as above. Then $m \notin j^{1}$, and the inclusion $\beta^{-}[h^{1}] \subseteq j^{1}$ yields $hR \beta(m)$. But $n \in \alpha[A]^{1}$ means $\alpha[A] \subseteq n^{1}$, i.e. $A \subseteq \alpha^{-}[n^{1}] \subseteq m^{1}$, whence $m \in A^{1}$ and so $\beta(m) \in \beta[A]^{1}$. This together with $hR \beta(m)$ gives $h \notin \beta[A]^{1}$, proving the equality $\alpha[A]^{1} = \beta[A]^{1}$.

**Proposition 3.2** The mapping pair $(\alpha, \beta)$ is concept continuous

iff $\alpha^{-}[C^{1}] = \beta^{-}[D^{1}]$ for all $C \subseteq H$ and $\beta^{-}[D^{1}] = \alpha^{-}[C^{1}]$ for all $D \subseteq N$

iff $\alpha^{-}[n^{1}] = \beta^{-}[n^{1}]$ for all $n \in N$ and $\beta^{-}[h^{1}] = \alpha^{-}[h^{1}]$ for all $h \in H$

iff $(\alpha, \beta)$ reflects incidence, $\alpha[\beta^{-}[n^{1}]] \subseteq n^{1}$ and $\beta[\alpha^{-}[h^{1}]] \subseteq h^{1}$

iff $\alpha(j)R \beta(m)$ \iff there is an $m \in M$ with $jI m$ and $n^{1} \subseteq \beta(m)^{1}$, and $hR \beta(m)$ \iff there is a $j \in J$ with $jI m$ and $h^{1} \subseteq \alpha(j)^{1}$.

**Proof.** If $(\alpha, \beta)$ is concept continuous then, observing that for arbitrary $C \subseteq H$ the pair $(C^{1}, C^{1})$ is a concept, we infer that $(\alpha^{-}[C^{1}], \beta^{-}[C^{1}])$ is a concept, too. Hence $\alpha^{-}[C^{1}] = \beta^{-}[C^{1}]$, and dually for $D \subseteq N$, $\beta^{-}[D^{1}] = \alpha^{-}[D^{1}]$.

The latter two equations entail that $(\alpha, \beta)$ reflects incidence (Lemma 3.5) and that $\alpha^{-}[n^{1}] = \beta^{-}[n^{1}]$ for all $n \in N$ and $\beta^{-}[h^{1}] = \alpha^{-}[h^{1}]$ for all $h \in H$. Conversely, $jI m$ and $n^{1} \subseteq \beta(m)^{1}$ imply $\alpha(j) \subseteq \beta(m)$ and, a fortiori, $\alpha(j)^{1} \subseteq \beta(m)^{1}$.

Assume in turn the validity of these equations. If $hR \beta(m)$ then $m$ is not in $\beta^{-}[h^{1}] = \alpha^{-}[h^{1}]$. Thus, there exists a $j \in J$ with $jI m$ and $\alpha(j) \subseteq h^{1}$, i.e. $h^{1} \subseteq \alpha(j)^{1}$. Dually, $\alpha(j)^{1} \subseteq \beta(m)^{1}$ for some $m \in M$.

Finally, suppose that the last two conditions are fulfilled. Then $(\alpha, \beta)$ reflects incidence (take $n = \beta(m)$). In order to show that for any concept $(C, D)$ of $(H, N, R)$, the ‘inverse image’ $(\alpha^{-}[C], \beta^{-}[D])$ is a concept of $(J, M, I)$, we have to verify the equations $\alpha^{-}[D^{1}] = \beta^{-}[D^{1}]$ and $\beta^{-}[C^{1}] = \alpha^{-}[C^{1}]$. If $\alpha(j) \notin D^{1}$ then $\alpha(j) \subseteq \beta(m)$ for some $n \in D$. By hypothesis, there is an $m \in M$ with $jI m$ and $n^{1} \subseteq \beta(m)^{1}$, whence $m \in \beta^{-}[n^{1}] \subseteq \beta^{-}[D^{1}] = \beta^{-}[D]$ and so $j \notin \beta^{-}[D^{1}]$. This proves $\beta^{-}[D^{1}] \subseteq \alpha^{-}[D^{1}]$, and the other inclusion $\alpha^{-}[D^{1}] = \alpha^{-}[C^{1}] \subseteq \beta^{-}[C^{1}] = \beta^{-}[D^{1}]$ follows from Lemma 3.5.

**Corollary 3.2** A mapping pair $(\alpha, \beta)$ is a dense quasi-embedding iff it is both conceptual and concept continuous.

This may be deduced directly from Propositions 3.1 and 3.2 but is also a consequence of the results in the next two sections.

Our final lemma shows that fullness or density of one partner in a conceptual pair implies the corresponding property of the other.

**Lemma 3.6** (1) If $(\alpha, \beta)$ is a quasi-embedding then $\alpha$ and $\beta$ are full.

(2) A conceptual pair $(\alpha, \beta)$ is a quasi-embedding iff $\alpha$ is full iff $\beta$ is full.

(3) A conceptual pair $(\alpha, \beta)$ is dense iff $\alpha$ is dense iff $\beta$ is dense.

Again, (1) and (2) suggest to replace the word ‘full’ with ‘faithful’. 

9
4 Complete Lattices as Contexts

We come now to the central part of our investigations, demonstrating that our choice of morphisms was the ‘right one’ from a categorical point of view. Each of the previously introduced classes of mapping pairs is closed under (componentwise) composition and may, therefore, serve as the morphism class of a category of contexts. Specifically, we have the following categories of contexts and complete lattices (see the comments after the diagrams):

Table 4.1

<table>
<thead>
<tr>
<th>contexts</th>
<th>morphisms</th>
<th>lattices</th>
<th>morphisms</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cp</td>
<td>mapping pairs</td>
<td>Clp</td>
<td>mapping pairs</td>
</tr>
<tr>
<td>Cip</td>
<td>incidence preserving pairs</td>
<td>Clip</td>
<td>isotone mapping pairs</td>
</tr>
<tr>
<td>Crip</td>
<td>incidence reflecting pairs</td>
<td>Clrp</td>
<td>order reflecting pairs</td>
</tr>
<tr>
<td>Cep</td>
<td>quasi-embeddings</td>
<td>Clep</td>
<td>order embedding pairs</td>
</tr>
<tr>
<td>Cep*</td>
<td>(separately) continuous</td>
<td>Clc&lt;sub&gt;∧&lt;/sub&gt;</td>
<td>join-meet preserving pairs</td>
</tr>
<tr>
<td>Cc</td>
<td>conceptual pairs</td>
<td>Clc</td>
<td>complete homomorphisms</td>
</tr>
<tr>
<td>Cc&lt;sup&gt;*&lt;/sup&gt;</td>
<td>concept continuous pairs</td>
<td>Clc&lt;sub&gt;−&lt;/sub&gt;</td>
<td>doubly residuated maps</td>
</tr>
<tr>
<td>Cd</td>
<td>dense conceptual pairs</td>
<td>Cld</td>
<td>dense compl. homomorphisms</td>
</tr>
<tr>
<td>Cd&lt;sup&gt;*&lt;/sup&gt;</td>
<td>dense concept continuous</td>
<td>Cld&lt;sup&gt;∗&lt;/sup&gt;</td>
<td>doubly residuated surjections</td>
</tr>
<tr>
<td>Ce</td>
<td>conceptual quasi-embeddings</td>
<td>Cle</td>
<td>complete embeddings</td>
</tr>
<tr>
<td>Ce&lt;sup&gt;*&lt;/sup&gt;</td>
<td>concept continuous</td>
<td>Cle&lt;sup&gt;∗&lt;/sup&gt;</td>
<td>doubly residuated embeddings</td>
</tr>
<tr>
<td>Cde</td>
<td>dense quasi-embeddings</td>
<td>Clde</td>
<td>dense complete embeddings</td>
</tr>
<tr>
<td>Ci</td>
<td>context isomorphisms</td>
<td>Cli</td>
<td>lattice isomorphisms</td>
</tr>
</tbody>
</table>
In view of the intended correspondences between contexts and complete lattices, we have included here a few less common types of morphisms: an isotope mapping pair \((\alpha, \beta)\) between complete lattices (or posets) is characterized by the implication \(x \leq y \Rightarrow \alpha(x) \leq \beta(y)\), while order reflecting pairs are characterized by the reverse implication, and order embedding pairs by the corresponding equivalence. By a join-meet preserving pair, we mean a mapping pair \((\alpha, \beta)\) between complete lattices such that \(\alpha\) preserves arbitrary joins and \(\beta\) preserves arbitrary meets.

Each category listed below \(\text{Clic}\), the category of complete lattices and join-meet preserving pairs, may be embedded in that category by obvious identifications: complete homomorphisms \(\varphi\) are identified with pairs \((\varphi, \varphi)\), doubly residuated maps \(\psi\) with pairs \((\psi, \psi^{**})\), and doubly residual maps \(\psi\) with pairs \((\psi_{*,*}, \psi)\). Thereby, the category \(\text{Clic}\) with complete homomorphisms is identified with that subcategory of \(\text{Clic}\) whose mapping pairs \((\alpha, \beta)\) have equal components \(\alpha = \beta\), whereas both \(\text{Clce}\), the category with doubly residuated morphisms, and \(\text{Clc}^*\), the category with doubly residual morphisms, are identified with that subcategory \(\text{Clic}^*\) of \(\text{Clic}\) whose morphisms \((\alpha, \beta)\) satisfy the equation \(\alpha^* = \beta^*\).

For any complete lattice \(L = (L, \leq)\), the complete or large context \(\mathcal{K}L = (L, \leq, \leq)\) is the greatest context whose concept lattice is isomorphic to \(L\). Thus, we have a functor \(\mathcal{K} : \text{Clic} \to \text{Ccp}\), sending \(L\) to \(\mathcal{K}L\) and acting identically on mapping pairs. Indeed, a map \(\alpha\) between complete lattices preserves arbitrary joins iff it is residuated, i.e. preimages of principal ideals are principal ideals, and these are just the extents of the associated contexts; and dually, a map preserves arbitrary meets iff it intent continuous. Hence, up to identification of complete lattices \(L\) with their complete contexts \(\mathcal{K}L\), the category \(\text{Clic}\) may be regarded as a full subcategory of \(\text{Ccp}\).

Note that by Lemma 3.2, a map \(\varphi\) between complete lattices is join dense (resp. meet dense) iff it is extent (resp. intent) dense as a map between the associated contexts. Furthermore, recall that a join- or meet-preserving dense map is already surjective; in particular, dense complete embeddings are already isomorphisms. In all, we have:

**Theorem 4.1** Up to canonical identifications, the functor \(\mathcal{K}\) fully embeds each of the categories of complete lattices on the right hand of Table 4.1 in the corresponding category of contexts on the left hand. In particular, associating with any complete homomorphism \(\varphi\) the pair \((\varphi, \varphi)\), one obtains full embeddings

- of \(\text{Clic}\) in \(\text{Cc}\), of \(\text{Cld}\) in \(\text{Cd}\), of \(\text{Cle}\) in \(\text{Ce}\), and of \(\text{Cli}\) in \(\text{Cde}\).

Similarly, associating with any doubly residuated map \(\psi\) the pair \((\psi, \psi^{**})\) and with any doubly residual map \(\psi\) the pair \((\psi_{*,*}, \psi)\), one obtains full embeddings

- of \(\text{Clce}\) and \(\text{Clc}^*\) in \(\text{Cc}_*\), of \(\text{Cld}_*\) and \(\text{Cld}^*\) in \(\text{Cd}_*\), etc.
Proof. It remains to verify the following facts about a mapping pair \((\alpha, \beta)\) between complete contexts \(K\) and \(K_L\):

1. \((\alpha, \beta)\) is conceptual iff \(\alpha = \beta\) is a complete homomorphism,
2. \((\alpha, \beta)\) is dense and conceptual iff \(\alpha = \beta\) is a surjective complete homomorphism,
3. \((\alpha, \beta)\) is a conceptual quasi-embedding iff \(\alpha = \beta\) is an injective complete homomorphism,
4. \((\alpha, \beta)\) is a dense quasi-embedding iff \(\alpha = \beta\) is an isomorphism,

\((1^\ast)\) \((\alpha, \beta)\) is concept continuous iff \(\alpha^{**} = \beta\) (hence \(\alpha = \beta_s\) is doubly residuated and \(\beta\) is doubly residual),

\((2^\ast)\) \((\alpha, \beta)\) is concept continuous and dense iff \(\alpha^{**} = \beta\) is surjective

if \(\alpha = \beta_s\) is surjective,

\((3^\ast)\) \((\alpha, \beta)\) is a concept continuous quasi-embedding iff \(\alpha^{**} = \beta\) is injective

if \(\alpha = \beta_s\) is injective.

Concerning (1), note first that for a conceptual morphism \((\alpha, \beta)\) between complete contexts, both \(\alpha\) and \(\beta\) are continuous, whence \(\alpha\) preserves joins and \(\beta\) preserves meets. By Proposition 3.1, we have \(\alpha(x)^\downarrow = \beta[x^\uparrow]\), which means \(\alpha(x) = \bigwedge \beta[x^\uparrow] = \beta(x)\), because \(\beta\) is isotone. Conversely, every complete homomorphism \(\varphi\) yields a conceptual morphism \((\varphi, \varphi)\) (cf. [6]).

The equivalences (2) and (3) are now immediate consequences of the remarks before the theorem. Concerning (4), recall the important fact that every dense quasi-embedding \((\alpha, \beta)\) is conceptual (and concept continuous), whence in the present situation \(\alpha = \beta\) is a join- and meet-dense complete embedding and consequently an isomorphism.

For \((1^\ast)\), suppose first that \((\alpha, \beta)\) is concept continuous. Then the equation \(\alpha^{-}[x^\uparrow] = \beta^{-}[x^\uparrow]\) (see Proposition 3.2) yields a map

\[\varphi : L \rightarrow K, \quad \varphi(x) = \bigvee \alpha^{-}[x^\uparrow] = \bigwedge \beta^{-}[x^\uparrow]\]

which is upper adjoint to \(\alpha\) and lower adjoint to \(\beta\), whence \(\alpha^{**} = \varphi^* = \beta\) and \(\alpha = \varphi_s = \beta_s\). Conversely, for a doubly residual map \(\psi : K \rightarrow L\), the lower adjoint \(\varphi = \psi_s\) is a complete homomorphism, and the pair \((\psi^{**}, \psi) = (\varphi_s, \varphi^*)\) is concept continuous, since each concept of \(KL\) has the form \((x^\uparrow, x^\downarrow)\), and consequently \((\psi^{**}[x^\uparrow], \psi^{-}[x^\downarrow]) = (\varphi(x)^\downarrow, \varphi(x)^\uparrow)\) is a concept of \(KK\).

The remaining statements are obtained as before, using the remarks on density and embedding properties of join- or meet-preserving maps. \(\Box\)

Note that the above claims remain valid if we replace join- and meet-preserving maps with residuated and residual maps between arbitrary posets. Thus the category of posets with residuated and residual maps is a full subcategory of the category \(Cc\), etc.
5 The Concept Lattice as a Covariant Functor

The point is now that the embedding functor $K$ has a left adjoint, sending on the object level each context to its concept lattice. Thus, for any context $K = (J, M, I)$ and $L = (H, N, R)$ a mapping pair $(\alpha^-, \beta^-)$ by

$$\alpha^- : B_K \rightarrow B_L, \quad \alpha^-(A, B) = (\alpha[A]^!, \alpha[A]^!),$$
$$\beta^- : B_K \rightarrow B_L, \quad \beta^-(A, B) = (\beta[B]^!, \beta[B]^!).$$

**Proposition 5.1** A mapping pair $(\alpha, \beta)$ between contexts $K$ and $L$ is
- separately continuous iff $\alpha^-$ preserves joins and $\beta^-$ meets,
- conceptual iff $\alpha^-=\beta^-$ is a complete homomorphism,
- dense conceptual iff $\alpha^-=\beta^-$ is a surjective complete homomorphism,
- a conceptual quasi-embedding iff $\alpha^-=\beta^-$ is a complete embedding,
- a dense quasi-embedding iff $\alpha^-=\beta^-$ is an isomorphism.

On the other hand, $(\alpha, \beta)$ is concept continuous
- iff $\alpha^-$ is doubly residuated with $\alpha^{---}=\beta^-$
- iff $\beta^-$ is doubly residual with $\alpha^{---}=\beta^{---}$. 

**Proof.** The first equivalence follows from Proposition 2.1. The second is obtained from the first one and Proposition 3.1. For the statements about density and embedding properties, apply Lemma 2.1.

In the case of a concept continuous pair $(\alpha, \beta)$, we have a well-defined complete homomorphism

$$\varphi : B_L \rightarrow B_K, \quad (C, D) \mapsto (\alpha^-[C], \beta^-[D])$$

because $\alpha^-$ and $\beta^-$ preserve arbitrary intersections. The equivalences

$$\alpha^-(A, B) \leq (C, D) \iff \alpha[A] \subseteq C \iff A \subseteq \alpha^-[C] \iff (A, B) \leq \varphi(C, D)$$

show that $\varphi$ is the upper adjoint of $\alpha^-$. A dual argument shows that $\varphi$ is the lower adjoint of $\beta^-$. Hence $\alpha^{---}=\varphi=\beta^{---}$.

Conversely, if $\alpha^{---}=\beta^-$ then $\varphi = \alpha^{---}$ is a complete homomorphism from $KL$ into $KK$ which is upper adjoint to $\alpha^-$ and lower adjoint to $\beta^-$. For any two concepts $(A, B) \in B_K$ and $(C, D) \in B_L$, we have the equivalences

$$A, B \leq \varphi(C, D) \iff \alpha^-(A, B) \leq (C, D) \iff (A, B) \leq (\alpha^-[C], \alpha^-[C]^!),$$
$$\varphi(C, D) \leq (A, B) \iff (C, D) \leq \beta^-(A, B) \iff (\beta^-[D], \beta^-[D]) \leq (A, B),$$

which amount to the equations

$$\varphi(C, D) = (\alpha^-[C], \beta^-[D]) = (\alpha^-[C], \alpha^-[C]^!) = (\beta^-[D]^!, \beta^-[D]),$$

establishing the concept continuity of $(\alpha, \beta)$. □
Basic in the study of concept lattices are the dense quasi-embeddings
\[ \eta_K : K \rightarrow KB = (B_K, B_K, \leq), \]
where
\[ \gamma_K : J \rightarrow B_K \quad \text{and} \quad \mu_K : M \rightarrow B_K \]
are the natural object and attribute insertion maps, respectively [10, 14]:
\[ \gamma(j) = (j^{\uparrow\downarrow}, j^{\uparrow}) \quad \text{and} \quad \mu(m) = (m^{\downarrow}, m^{\downarrow\uparrow}). \]
It was pointed out in the preprint version of [6] that the quasi-embeddings \( \eta_K \) are the reflection morphisms for a reflector \( KB \) from the category \( Cc \) of contexts and conceptual mapping pairs to the full subcategory \( Ccc \) of complete contexts and complete homomorphisms. The construction from [6] may be extended, in a straightforward manner, to the categories \( Ccp \) and \( Clc \): for any separately continuous mapping pair \((\alpha, \beta)\) between contexts \( K \) and \( L \), we put \( B(\alpha, \beta) = (\alpha \rightarrow, \beta \rightarrow) \). In case \((\alpha, \beta)\) is conceptual, we have \( \alpha \rightarrow = \beta \rightarrow \), so that we may replace \((\alpha \rightarrow, \beta \rightarrow)\) with \( B(\alpha, \beta) = \alpha \rightarrow \). Similarly, if \((\alpha, \beta)\) is concept continuous, the second component is determined by the first one via \( \alpha^* \beta = \beta^* \), so that it is more convenient to put \( B(\alpha, \beta) = \alpha \rightarrow \) (or, alternately, \( B(\alpha, \beta) = \beta \rightarrow \)). Concerning adjoint functors, we refer to [1].

**Theorem 5.1** \( B \) is a covariant functor

- from \( Ccp \) to \( Clc \)
- from \( Cc \) to \( Clc \)
- from \( Cc^* \) to \( Clc^* \) and \( Clc^* \)
- from \( Cd \) to \( Cld \)
- from \( Cd^* \) to \( Cld^* \) and \( Cld^* \)
- from \( Ce \) to \( Cle \)
- from \( Ce^* \) to \( Cle^* \) and \( Cle^* \)

Furthermore, \( B \) is left adjoint to the functor \( K \) in the opposite direction. The unit of the adjunctions is \( \eta \), and the counit is an isomorphism.

**Proof.** The functor properties of \( B \) are easily checked, using Proposition 2.1 and Proposition 5.1. Concerning adjointness, we have to show that for every complete lattice \( L = (L, \leq) \) and for every separately continuous morphism \((\alpha, \beta)\) from an arbitrary context \( K = (J, M, I) \) into \( KL = (L, L, \leq) \), there is a unique pair \((\alpha^\vee, \beta^\wedge)\) such that
\[ (\alpha, \beta) = (\alpha^\vee, \beta^\wedge) \circ \eta_K, \quad \text{i.e.} \quad \alpha = \alpha^\vee \circ \gamma_K \quad \text{and} \quad \beta = \beta^\wedge \circ \mu_K. \]
Defining for each concept \((A, B) \in B_K \)
\[ \alpha^\vee(A, B) = \vee A[B] \quad \text{and} \quad \beta^\wedge(A, B) = \wedge B[B], \]
we see that \( \alpha^\vee \) is lower adjoint to the map
\[ \alpha^\vee_x : L \rightarrow B_K, \quad x \mapsto (\alpha^{-}[x], \alpha^{-}[\downarrow x]), \]
and that \( \beta^\wedge \) is upper adjoint to the map
\[ \beta^\wedge_x : L \rightarrow B_K, \quad x \mapsto (\beta^{-}[\uparrow x], \beta^{-}[\uparrow x]). \]
Hence, $\alpha^\vee$ preserves joins, $\beta^\wedge$ preserves meets, and
\[
\alpha^\vee \circ \gamma(j) = \bigvee \alpha[j_{1}^{\downarrow}] = \alpha(j) \quad \text{by continuity of } \alpha,
\]
\[
\beta^\wedge \circ \mu(m) = \bigwedge \beta[m_{1}^{\uparrow}] = \beta(m) \quad \text{by continuity of } \beta.
\]
The uniqueness of a join-meet preserving pair $(\alpha^\vee, \beta^\wedge)$ with $\alpha = \alpha^\vee \circ \gamma_K$ and $\beta = \beta^\wedge \circ \mu_K$ follows from join-density of $\gamma_K$ and meet-density of $\mu_K$.

Now, if $(\alpha, \beta) : K \to L$ is an arbitrary Cep-morphism (that is, a separately continuous mapping pair) then there is a unique Clc$^\vee$-morphism $(\varphi, \psi) : B_K \to BL$ satisfying the identity $(\varphi, \psi) \circ \eta_K = \eta_L \circ (\alpha, \beta)$, namely
\[
\varphi(A, B) = \bigvee \gamma_L[\alpha[A]] = (\alpha[A]^\downarrow, \alpha[A]^\uparrow) = \alpha^{-}(A, B),
\]
\[
\psi(A, B) = \bigwedge \mu_L[\beta[B]] = (\beta[B]^\uparrow, \beta[B]^\downarrow) = \beta^{-}(A, B),
\]
and consequently $(\varphi, \psi) = \mathcal{B}(\alpha, \beta)$. This shows that $\mathcal{B}$ is in fact left adjoint to $K$, and that $\eta$ is the unit of the adjunction. The counit $\varepsilon$ is constituted by the natural isomorphisms
\[
\varepsilon_L : \mathcal{B}K \to L \quad \text{with} \quad \varepsilon_L(\uparrow x, \downarrow x) = x. \quad \square
\]

Invoking Proposition 2.1 once more, we conclude:

**Corollary 5.1** A mapping pair $(\alpha, \beta) : K \to L$ is separately continuous iff there exists a (unique) join-meet preserving pair $(\varphi, \psi) : KB_K \to KB\mathcal{L}$ with
\[
\varphi \circ \gamma_K = \gamma_L \circ \alpha \quad \text{and} \quad \psi \circ \mu_K = \mu_L \circ \beta,
\]
namely $(\varphi, \psi) = (\alpha^{-}, \beta^{-})$. Furthermore,
- $(\alpha, \beta)$ is conceptual iff $\varphi = \psi$ is a complete homomorphism,
- $(\alpha, \beta)$ is concept continuous iff $\varphi^* = \psi^*$ is a complete homomorphism,
- $(\alpha, \beta)$ is a dense quasi-embedding iff $\varphi = \psi (= \psi^{**})$ is an isomorphism.

**Corollary 5.2** Up to identification between complete lattices and complete contexts, we have full reflective subcategories
\[
\text{Clc}^\vee \hookrightarrow \text{Cep}, \quad \text{Cle} \hookrightarrow \text{Cc}, \quad \text{Cld} \hookrightarrow \text{Cd}, \quad \text{Cle} \hookrightarrow \text{Ce},
\]
\[
\text{Cli} \hookrightarrow \text{Cde}, \quad \text{Clc}_e \hookrightarrow \text{Cc}_e, \quad \text{Cld}_e \hookrightarrow \text{Cd}_e, \quad \text{Cle}_e \hookrightarrow \text{Ce}_e,
\]
and analogous reflections for the corresponding categories with doubly residual morphisms.
Corollary 5.3 For all contexts $\mathbb{K}$ and complete lattices $L$, a mapping pair $(\alpha, \beta): \mathbb{K} \to KL$ is conceptual iff there is a unique complete homomorphism $\varphi: B\mathbb{K} \to L$ such that $(\alpha, \beta) = (\varphi, \varphi) \circ \eta_{\mathbb{K}}$, i.e. $\alpha = \varphi \circ \gamma_{\mathbb{K}}$ and $\beta = \varphi \circ \mu_{\mathbb{K}}$. Moreover, $(\alpha, \beta)$ is a dense quasi-embedding iff $\varphi$ is an isomorphism.

These results extend known facts about the Dedekind-MacNeille completion (by cuts) of ordered sets (cf. [3, 6]). Another immediate application is the Fundamental Theorem on Concept Lattices (see e.g. [14] or [10]), saying that a concept lattice $B(J, M, I)$ is isomorphic to a given complete lattice $L$ iff there exists a join-dense map $\gamma$ from $J$ into $L$ and a meet-dense map $\mu$ from $M$ into $L$ such that $jIm \iff \gamma(j) \leq \mu(m)$.

If $(J, M, I)$ is a subcontext of a context $(H, N, R)$ (that is, $J \subseteq H$, $M \subseteq N$ and $I = (J \times M) \cap R$), we may consider the respective inclusion maps.

Corollary 5.4 For a subcontext $\mathbb{K} = (J, M, I)$ of a context $L = (H, N, R)$, the following conditions are equivalent:

(a) The inclusion maps $J \hookrightarrow H$ and $M \hookrightarrow N$ form a conceptual pair.
(b) $(A^R \cap M)_R \subseteq A^R$ for $A \subseteq J$ and $(B_R \cap J)_R \subseteq B^R$ for $B \subseteq M$.
(c) If $hRn$ then there are $jIm$ with $h^R \cap M \subseteq j^R$ and $nR \cap J \subseteq m_R$.
(d) $(A, B) \mapsto (A^R, A^R) = (B_R, B^R)$ is a complete homomorphism from $B\mathbb{K}$ to $B\mathbb{L}$.
(e) There is a complete homomorphism $\varphi: B\mathbb{K} \to B\mathbb{L}$ with $\varphi \circ \eta_{\mathbb{K}} = \eta_{\mathbb{L}}$.

Similarly, taking for $\alpha$ and $\beta$ identity maps but different incidence relations, we arrive at

Corollary 5.5 For two contexts $\mathbb{K} = (J, M, I)$ and $\mathbb{L} = (J, M, R)$ with the same underlying sets, the following conditions are equivalent:

(a) The identity pair $(id_G, id_M)$ is conceptual.
(b) $A^R = A^R$ for $A \subseteq G$ and $B_R = B^R$ for $B \subseteq M$.
(c) $I \subseteq R$, and for all $hRn$ there are $jIm$ with $h^R \subseteq j^I$ and $nR \subseteq m_I$.
(d), (e) As in Corollary 5.4.

Note that in contrast to Corollary 5.4, the complete homomorphisms in Corollary 5.5 are always surjective, because identity maps are trivially dense (but not necessarily full).
6 The Concept Lattice as a Contravariant Functor

We have already seen in the preceding section that every concept continuous context morphism \((\alpha, \beta) : K \rightarrow L\) gives rise to a complete homomorphism in the opposite direction,

\[ \varphi : \mathcal{B}L \rightarrow \mathcal{B}K, \ (C, D) \mapsto (\alpha^-[C], \beta^-[D]). \]

More generally, any separately continuous context morphism \((\alpha, \beta) : K \rightarrow L\) induces a meet-preserving map \(\alpha^- : \mathcal{B}L \rightarrow \mathcal{B}K, \ (C, D) \mapsto (\alpha^-[C], \alpha^-[C]^\perp)\) and a join-preserving map \(\beta^- : \mathcal{B}L \rightarrow \mathcal{B}K, \ (C, D) \mapsto (\beta^-[D]^\perp, \beta^-[D])\), and these two maps coincide iff \((\alpha, \beta)\) is concept continuous.

The category \(\text{Clc}^\wedge\) of complete lattices and meet-join preserving pairs \((\psi, \varphi)\) (where \(\psi : K \rightarrow L\) preserves arbitrary meets and \(\varphi : K \rightarrow L\) preserves arbitrary joins) is isomorphic to the category \(\text{Clc}^\wedge\) by means of two essentially different functors: one of them exchanges the first and the second component in the mapping pairs, while the other keeps the morphisms fixed and reverses the lattice orders. But there is also a dual isomorphism between \(\text{Clc}^\wedge\) and \(\text{Clc}^\vee\), obtained by passing to the order-theoretical adjoints:

\[ G : \text{Clc}^\wedge \rightarrow \text{Clc}^\vee, \ G_L = L, \ G(\varphi, \psi) = (\varphi^*, \psi_*), \]

\[ H : \text{Clc}^\vee \rightarrow \text{Clc}^\wedge, \ H_L = L, \ H(\psi, \varphi) = (\psi_*, \varphi^*). \]

Obviously, these mutually inverse functors induce dual isomorphisms between the categories \(\text{Clc}\) and \(\text{Clc}^\vee\), etc.

Now, by our previous considerations, we have a contravariant functor

\[ \mathcal{B}^- : \text{Ccp} \rightarrow \text{Clc}^\wedge, \ \mathcal{B}^-K = \mathcal{B}K, \ \mathcal{B}^-(\alpha, \beta) = (\alpha^-, \beta^-), \]

and also a contravariant functor in the other direction, namely

\[ \mathcal{K}^- : \text{Clc}^\wedge \rightarrow \text{Ccp}, \ \mathcal{K}^-L = \mathcal{K}L, \ \mathcal{K}^-(\psi, \varphi) = (\psi_*, \varphi^*). \]

Moreover, these functors are linked with the covariant functors \(\mathcal{B}\) and \(\mathcal{K}\) by the identities

\[ \mathcal{B}^- = G \circ \mathcal{B}, \ \mathcal{B} = H \circ \mathcal{B}^-, \ \mathcal{K}^- = K \circ H, \ \mathcal{K} = K^- \circ G. \]

Therefore, the adjunction in Theorem 5.1 turns into a dual adjunction for the corresponding contravariant functors:

**Theorem 6.1** The contravariant functor \(\mathcal{B}^- : \text{Ccp} \rightarrow \text{Clc}^\wedge\) is dually adjoint to the contravariant functor \(\mathcal{K}^- : \text{Clc}^\wedge \rightarrow \text{Ccp}\). Furthermore, these functors induce dual adjunctions between \(\text{Cc}\) and \(\text{Clc}^\vee\), \(\text{Cc}^\vee\) and \(\text{Clc}\), \(\text{Cde}\) and \(\text{Cli}\), etc.
Corollary 6.1 For any contexts $\mathbb{K}$ and any complete lattice $L$, a mapping pair $(\alpha, \beta) : \mathbb{K} \rightarrow \mathbb{KL}$ is concept continuous iff there is a unique complete homomorphism $\varphi : L \rightarrow B\mathbb{K}$ such that $\varphi \circ \gamma_{\mathbb{K}} = \alpha$ and $\varphi^* \circ \mu_{\mathbb{K}} = \beta$.

\[\eta_{\mathbb{K}} = (\gamma_{\mathbb{K}}, \mu_{\mathbb{K}})\]

Corollary 6.2 A mapping pair $(\alpha, \beta)$ between contexts $\mathbb{K}$ and $L$ is concept continuous iff there exists a (unique) complete homomorphism $\varphi : B\mathbb{L} \rightarrow B\mathbb{K}$ such that $\varphi^* \circ \gamma_{\mathbb{L}} = \gamma_{\mathbb{L}} \circ \alpha$ and $\varphi^* \circ \mu_{\mathbb{L}} = \mu_{\mathbb{L}} \circ \beta$, namely $\varphi = \alpha^{-} = \beta^{-}$.

Corollary 6.3 For a subcontext $\mathbb{K} = (J,M,I)$ of a context $L = (H,N,R)$, the following conditions are equivalent:

(a) The inclusion morphism from $\mathbb{K}$ into $L$ is concept continuous.

(b) $\mathbb{K}$ is compatible, that is, $(C^R \cap M)_R \cap J \subseteq C^R$ for all $C \subseteq H$ and $(D_J \cap J)_R \cap M \subseteq D^R_R$ for all $D \subseteq N$.

(c) For all $j \in J$ and $n \in N \setminus j^R$ there is an $m \in M \setminus j^R$ with $n_R \subseteq m_R$, and for all $m \in M$ and $h \in H \setminus m_R$ there is a $j \in G \setminus m_R$ with $h^R \subseteq j^R$.

(d) The trace map $\varphi : B\mathbb{L} \rightarrow B\mathbb{K}$, $(C,D) \mapsto (C \cap G, D \cap M)$ is a complete homomorphism.

(e) There exists a unique complete homomorphism $\varphi$ from $B\mathbb{L}$ onto $B\mathbb{K}$ with $\varphi^* \circ \gamma_{\mathbb{L}} = \gamma_{\mathbb{L}}$ and $\varphi^* \circ \mu_{\mathbb{L}} = \mu_{\mathbb{L}}$.

Another frequently used application of our results on concept continuous maps is obtained by taking for $\alpha$ and $\beta$ identity maps.

Corollary 6.4 For contexts $\mathbb{K} = (G,M,I)$ and $L = (G,M,R)$, the following conditions are equivalent:

(a) The identity pair $(\text{id}_G, \text{id}_M) : \mathbb{K} \rightarrow \mathbb{L}$ is concept continuous.

(b) $R \subseteq I$, $A^R_I \subseteq A^R_R$ for $A \subseteq G$, and $B^R_R \subseteq B^R_R$ for $B \subseteq M$.

(c) $R \subseteq I$, and $(j,m) \in I \setminus R$ implies $h \mid m$ for some $h \in G$ with $j^R \subseteq h^R$ and $j \mid n$ for some $n \in M$ with $m_R \subseteq n_R$.

(d) $R$ is a closed relation of $\mathbb{K}$, that is, $R \subseteq G \times M$ and $B\mathbb{L} \subseteq B\mathbb{K}$.

(e) $B\mathbb{L}$ is a complete sublattice of $B\mathbb{K}$.

Parts of the last two corollaries have been discovered earlier by Ganter, Wille and Reuter (see [15, 16] and [13]) and were included in [10].
7 Purified Contexts and Doubly Based Lattices

In order to obtain not only (dual) adjunctions but even (dual) equivalences between certain categories of contexts and complete lattices, we must transfer the role played by the sets of objects and attributes, respectively, to the realm of complete lattices. To that aim, we introduce doubly based lattices as triples \((K, J, M)\) where \(K = (K, \leq)\) is a complete lattice, \(J\) is a join-dense subset (join-base) and \(M\) is a meet-dense subset (meet-base) of \(K\) (see [9]). Any context \(K = (J, M, I)\) gives rise to a doubly based lattice

\[ B^oK = (B^K, J_0, M_0) \text{ with } J_0 = \gamma_K[J] \text{ and } M_0 = \mu_K[M]. \]

For any separately continuous morphism \((\alpha, \beta) : K = (J, M, I) \to L = (H, N, R)\) the lifted map \(\alpha^-\) preserves not only joins but also the join-bases, i.e. \(\alpha^-[J_0] \subseteq H_0\), and \(\beta^-\) preserves not only meets but also the meet-bases, i.e. \(\beta^-[M_0] \subseteq N_0\), on account of the equations

\[ \alpha^- \circ \gamma_K = \gamma_L \circ \alpha \text{ and } \beta^- \circ \mu_K = \mu_L \circ \beta. \]

Putting \(B^o(\alpha, \beta) = (\alpha^-, \beta^-)\), we obtain a functor \(B^o\) from the category \(\text{Ccp}\) of contexts with separately continuous morphisms to the category \(\text{Clb}^\cap \cap\) of doubly based lattices with mapping pairs \((\varphi, \psi)\) such that \(\varphi\) preserves joins and the selected join-bases, while \(\psi\) preserves meets and the selected meet-bases. In the opposite direction, we may assign to each doubly based lattice \(K = (K, J, M)\) the base context \(K^oK = (J, M, \leq)\) where \(\leq\) denotes the relation between \(J\) and \(M\) induced by the given order relation of \(K\). Then \(K^o\) becomes a functor, acting on morphisms by restriction to the given join- and meet-bases (see the proof of Theorem 7.1). A context \(K\) is said to be purified if \(\gamma_K\) and \(\mu_K\) are injective – in other words, if \(\eta_K\) induces an isomorphism between \(K\) and the context \(K^oB^oK = (J_0, M_0, \leq)\). For any doubly based lattice \(K = (K, J, M)\), the context \(K^oK = (J, M, \leq)\) is purified. Moreover, by join-density of \(J\) and meet-density of \(M\), we have an isomorphism

\[ \varepsilon_K : B(J, M, \leq) \to K, \; (A, B) \mapsto \bigvee A = \bigwedge B \]

sending \(J_0\) to \(J\) and \(M_0\) to \(M\), with inverse

\[ \iota_K : K \to B(J, M, \leq), \; x \mapsto (J \cap \downarrow x, M \cap \uparrow x). \]

Since \(\iota_K\) maps \(J\) onto \(J_0\) and \(M\) onto \(M_0\), we may regard that isomorphism as a morphism between \(K\) and \(B^oK^oK\) in the category \(\text{Clb}^\cap\).

**Theorem 7.1** The augmented concept lattice functor \(B^o\) induces an equivalence between the category \(\text{Ccp}^o\) of purified contexts with separately continuous morphisms and the category \(\text{Clb}^\cap\) of doubly based lattices with join-meet preserving pairs that induce mappings between the respective bases. In the opposite direction, an equivalence is established by the functor \(K^o\).

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Similarly, the category $\mathbf{C}^o$ of purified contexts with conceptual morphisms is equivalent to the category $\mathbf{C}y^o$ of doubly based lattices with complete homomorphisms preserving the join- and meet-bases.

Proof. We have already seen that for any $\mathbf{Cc}^o$-morphism $(\alpha, \beta)$, the map $B^o(\alpha, \beta) = (\alpha^\rightarrow, \beta^\rightarrow)$ is a $\mathbf{Clb}^\hat{\diamond}$-morphism. The equation

$$K^oB^o(\alpha, \beta) \circ \eta_K = (\alpha^\rightarrow \circ \gamma_K, \beta^\rightarrow \circ \mu_K) = (\gamma_L \circ \alpha, \mu_L \circ \beta) = \eta_L \circ (\alpha, \beta)$$

shows that $\eta$ is a natural isomorphism between the identity functor on the category of purified contexts and the composite functor $K^oB^o$.

On the other hand, given an arbitrary $\mathbf{Clb}^\hat{\diamond}$-morphism $(\varphi, \psi)$ between doubly based lattices $K = (K, J, M)$ and $L = (L, H, N)$, we must show that $\alpha : J \to H$, $j \mapsto \varphi(j)$ and $\beta : M \to N$, $m \mapsto \psi(m)$ are continuous maps. The extents of $K^oL$ are all sets of the form $N \cap \downarrow y$, and

$$\alpha^-[N \cap \downarrow y] = \{ j \in J : \varphi(j) \leq y \} = \{ j \in J : j \leq \varphi^*(y) \} = J \cap \downarrow \varphi^*(y)$$

is then an extent, too. Thus, $\alpha$ is extent continuous, and dually, $\beta$ is intent continuous, so that $(\alpha, \beta)$ is a $\mathbf{C}^o$-morphism. Moreover, the equations

$$H \cap \downarrow \varphi(x) = H \cap \downarrow \varphi[ J \cap \downarrow x ] = \varphi[J \cap \downarrow x]$$

and

$$N \cap \uparrow \psi(x) = N \cap \uparrow \psi[M \cap \uparrow x] = \psi[M \cap \uparrow x]$$

yield the naturality of the isomorphism $\iota : 1_{\mathbf{Clb}^\hat{\diamond}} \to B^oK^o$ and its inverse:

$$\iota_L \circ (\varphi, \psi) = B^oK^o(\varphi, \psi) \circ \iota_K, \text{ hence } (\varphi, \psi) \circ \varepsilon_K = \varepsilon_L \circ B^oK^o(\varphi, \psi).$$

If $\varphi = \psi$ is a $\mathbf{Cl}^o$-morphism, we have to verify that the mapping pair $K^o(\varphi, \varphi) = (\alpha, \beta)$ from $K^oK$ to $K^oL$ is conceptual. To that aim, we compute for $A \subseteq J$ and for $y = \uparrow \varphi[A]$, using join- and meet-density of the bases:

$$y = \varphi(\downarrow A) = \varphi(\wedge(M \cap \downarrow A)) = \wedge \varphi[M \cap \downarrow A] = \wedge \varphi[A],$$

$$\alpha[A] = \varphi[A] = H \cap \downarrow y = \beta[A].$$

A dual reasoning yields the identity $\beta[B] = \alpha[B]$ for $B \subseteq M$.

Has the above equivalence theorem an analogue for concept continuous maps? The answer is in the affirmative, but the choice of morphisms is a bit subtle. Let us consider the category $\mathbf{Cl}_{\diamond\sharp}$ of doubly based lattices with join-base preserving doubly residuated maps $\psi$ whose double upper adjoint $\psi^{**}$ preserves the meet-bases. By passing to these double upper adjoints, one obtains an isomorphism between $\mathbf{Cl}_{\diamond\sharp}$ and the category $\mathbf{Cl}^\diamond\hat{\flat}$ of doubly based lattices with meet-base preserving doubly residual maps whose double lower adjoint preserves the join-bases. A different isomorphism between these categories results from dualization of the order relations, leaving unchanged the underlying mappings of the morphisms. But notice that $\mathbf{Cl}_{\diamond\sharp}$ and $\mathbf{Cl}^\diamond\hat{\flat}$ are not dual to the category $\mathbf{Cl}^o$, but to that category $\mathbf{Cl}_{\diamond}$ whose morphisms have a lower adjoint that preserves join-bases and an upper adjoint that preserves meet-bases.
Theorem 7.2 The functor $\mathbb{B}^\ast$, assigning to each concept continuous morphism $(\alpha, \beta)$ between purified contexts $\mathbb{K}$ and $\mathbb{L}$ the map $\alpha^- : \mathbb{B}\mathbb{K} \to \mathbb{B}\mathbb{L}$, yields an equivalence between the category $\mathbb{C}_o$ of purified contexts with concept continuous pairs and the category $\mathbb{C}^\ast$. Similarly, the functor $\mathbb{B}_o^\ast$, sending $(\alpha, \beta)$ to $\beta^-$, yields an equivalence between the categories $\mathbb{C}_o$ and $\mathbb{C}_o^\ast$. In the opposite direction, equivalences between $\mathbb{C}^\ast$ resp. $\mathbb{C}_o^\ast$ and $\mathbb{C}_o$ are given by restriction of the doubly residuated maps to the join-bases and of the doubly residual maps to the meet-bases, respectively.

Proof. We have seen earlier that for any concept continuous pair $(\alpha, \beta)$, the map $\alpha^-$ is doubly residuated, with $\alpha^- = \alpha^- = \beta^- = \beta^-$, and that $\alpha^-$ preserves the join-bases, while $\beta^-$ preserves the meet-bases.

Given a $\mathbb{C}^\ast$-morphism $\psi = \varphi : \mathbb{K} \to \mathbb{L}$, we show that the mapping $\langle \alpha, \beta \rangle_{\mathbb{K} \to \mathbb{L}}$ from $\mathbb{K} \to \mathbb{L}$ with $\alpha = \varphi|_J : J \to H$ and $\beta = \psi|_M : M \to N$ is concept continuous. For $C \subseteq H$, we have:

\begin{align*}
\psi(j) \in \alpha^- [C^\uparrow] & \iff \alpha(j) \in H \cap \downarrow \psi(C) \\
\iff \forall m \in M (\varphi(C) \leq m \Rightarrow j \leq m) \\
\iff \forall m \in M (\varphi^*(m) \in C \uparrow \Rightarrow j \leq m) \\
\iff \forall m \in M (\beta(m) \subseteq C^\uparrow \Rightarrow j \leq m) \iff j \in \beta^- [C^\uparrow].
\end{align*}

This and the dual equation $\beta^- [D^\downarrow] = \alpha^- [D^\downarrow]$ prove concept continuity.

That $\eta$ and $\varepsilon$ are natural isomorphisms between the identity functors and the functors composed by $\mathbb{K}_o$ and $\mathbb{B}_o$ is checked as before.

Corollary 7.1 The isomorphism classes of purified contexts bijectively correspond to the isomorphism classes of doubly based lattices. The purified contexts are exactly those of the form $\mathbb{K}^\circ \mathbb{K}$ for doubly based lattices $\mathbb{K}$, and the conceptual morphisms between purified contexts $\mathbb{K} = \mathbb{K}^\circ \mathbb{K}$ and $\mathbb{L} = \mathbb{K}^\circ \mathbb{L}$ are exactly the restrictions of the base-preserving complete homomorphisms between the doubly based lattices $\mathbb{K} \simeq \mathbb{B}^\circ \mathbb{K}$ and $\mathbb{L} \simeq \mathbb{B}^\circ \mathbb{L}$.

Conversely, every base-preserving complete homomorphism between doubly based lattices is induced by a unique conceptual morphism between the base contexts.

Corollary 7.2 The concept continuous morphisms between purified contexts $\mathbb{K} = \mathbb{K}^\circ \mathbb{K}$ and $\mathbb{L} = \mathbb{K}^\circ \mathbb{L}$ are exactly the pairs formed by the base restrictions of the lower adjoint $\varphi_*$ and the upper adjoint $\varphi^*$ of complete homomorphisms $\varphi$ from $\mathbb{L} \simeq \mathbb{B}^\circ \mathbb{L}$ to $\mathbb{K} \simeq \mathbb{B}^\circ \mathbb{K}$ such that $\varphi_*$ preserves the join-bases and $\varphi^*$ the meet-bases.

Conversely, every complete homomorphism having a join-base preserving lower adjoint and a meet-base preserving upper adjoint is induced by a unique concept continuous morphism in the opposite direction.
8 Dualities and Galois Connections for Contexts

As expected, there are not only equivalences but also dualities between certain categories of contexts and complete lattices. In most cases, such dualities are obtained by composing the already established equivalences with the dual isomorphisms, passing between lower and upper adjoints. For example, the category $\mathbf{Cl}^\circ$ of doubly based lattices with base-preserving complete homomorphisms is dually isomorphic to the following two categories with the same objects: the morphisms of $\mathbf{Cl}^{\circ\ast}$ are those doubly residual maps whose lower adjoint preserves join- and meet-bases (hence is a $\mathbf{Cl}^\circ$-morphism), and the morphisms in $\mathbf{Cl}_c^\ast$ are those doubly residuated maps whose upper adjoint preserves join- and meet-bases. Thus, Theorem 7.1 amounts to:

**Corollary 8.1** Sending each conceptual morphism $(\alpha, \beta)$ to the doubly residual map $\alpha^\leftarrow$, one obtains a dual equivalence between the category $\mathbf{C}^\circ$ of purified contexts with conceptual morphisms and the category $\mathbf{Cl}^{\circ\ast}$, while sending $(\alpha, \beta)$ to the doubly residuated map $\beta^\leftarrow$ yields a dual equivalence between the categories $\mathbf{C}^\circ$ and $\mathbf{Cl}_c^\ast$.

As already observed earlier, in the same way, the category $\mathbf{Cl}_c^\circ$ of doubly based lattices with complete homomorphisms $\varphi$ having a join-base preserving lower adjoint $\varphi^\ast$ and a meet-base preserving upper adjoint $\varphi^\circ$ is dually isomorphic to the categories $\mathbf{Cl}_c^{\circ\ast}$ and $\mathbf{Cl}_c^\ast$. Thus, from Theorem 7.2, we immediately derive:

**Corollary 8.2** Sending each concept continuous pair $(\alpha, \beta)$ to the complete homomorphism $\alpha^\leftarrow = \beta^\leftarrow$, one obtains a dual equivalence between the category $\mathbf{C}^\circ$ of purified contexts with concept continuous morphisms and the category $\mathbf{Cl}_c^\circ$.

A basic remark about canonical order structures on contexts is now long overdue. Both the objects and the attributes of any context carry a natural ‘specialization order’ (cf. the introduction), given by

\[ j \leq k \iff k^\uparrow \subseteq j^\uparrow \quad \text{and} \quad m \leq n \iff m^\downarrow \subseteq n^\downarrow. \]

These two relations are obviously quasi-orders (reflexive and transitive), and they are partial orders (antisymmetric) iff the context is purified. The restriction to purified contexts has great structural advantages and causes no essential loss of generality, because every context $K = (J, M, I)$ has a purification $K^\circ B^\circ K = (J_0, M_0, \leq)$ whose concept lattice is isomorphic to that of the original context. Note the following implication:

\[ j \leq k \, m \leq n \implies j \, I \, n. \]

It is also convenient to know that for any base context $K^\circ K = (J, M, \leq)$, the specialization orders are induced by the lattice order.
The following identities connecting the partners of mapping pairs between purified contexts are easily verified with the help of Propositions 3.1 and 3.2 (maxima and minima refer to the specialization orders):

**Proposition 8.1** Each partner of a conceptual mapping pair \((\alpha, \beta)\) between purified contexts determines the other uniquely, by the identities

\[
\alpha(j) = \max \{ h : j^\uparrow \subseteq \beta^{-}[h^\uparrow] \}, \quad \beta(m) = \min \{ n : m^\downarrow \subseteq \alpha^{-}[n^\downarrow] \}.
\]

Similarly, each partner of a concept continuous mapping pair \((\alpha, \beta)\) between purified contexts determines the other uniquely, by the identities

\[
\alpha(j) = \min \{ h : \beta^{-}[h^\uparrow] \subseteq j^\uparrow \}, \quad \beta(m) = \max \{ n : \alpha^{-}[n^\downarrow] \subseteq m^\downarrow \}.
\]

On account of these facts, it would suffice to consider single maps between the object sets or the attribute sets of purified contexts as conceptual or concept continuous morphisms. However, the approach via mapping pairs makes the interplay between objects and attributes of contexts more transparent.

In view of the striking similarities between conceptual and concept continuous morphisms, and encouraged by the dual isomorphisms between the corresponding lattice categories \(\text{Clc}\) and \(\text{Clc}^\ast\), etc. one might wish to find similar dualities between suitable subcategories of the purified context categories \(\text{C}_r \subseteq \text{Cc}\) and \(\text{C}_\circ \subseteq \text{Cc}^\ast\). This is in fact possible, but the appropriate choice of morphisms looks a bit technical at first glance. However, from the Galois-theoretical point of view, it is rather natural. By slight abuse of language, we call a mapping pair \((\alpha, \beta)\) between purified contexts \(K\) and \(L\) residuated if it is concept continuous, \(\alpha\) is residuated and \(\beta\) is dually residuated (i.e. residual) with respect to the specialization orders. On the other hand, we say \((\alpha, \beta)\) is residual if it is conceptual, \(\alpha\) is residual and \(\beta\) is dually residual (i.e. residuated). The resulting categories of purified contexts are denoted by \(\text{C}_r\) and \(\text{C}_\circ\), respectively. As morphisms in the corresponding category \(\text{Clr}\) of doubly based lattices we take the base-preserving and \(\ast\)-reflecting complete homomorphisms, i.e. those maps which do not only preserve join-bases and meet-bases, but also have lower adjoints preserving the join-bases and upper adjoints preserving the meet-bases.

**Lemma 8.1** If \((\alpha, \beta) : K \rightarrow L\) is a mapping pair between contexts such that \(\alpha : J \rightarrow H\) has an upper adjoint \(\alpha^\ast\) and \(\beta : M \rightarrow N\) has a lower adjoint \(\beta_\ast\), then the following statements are equivalent:

(a) \((\alpha, \beta)\) is concept continuous (residuated).

(b) \((\alpha^\ast, \beta_\ast)\) is conceptual (residual).

(c) \(\alpha(j)^\uparrow = \beta^{-}[j^\uparrow]\) for all \(j \in J\) and \(\beta(m)^\downarrow = \alpha^\ast^{-}[m^\downarrow]\) for all \(m \in M\).

(d) \(\alpha^\ast(h)^\uparrow = \beta^{-}[h^\uparrow]\) for all \(h \in H\) and \(\beta_\ast(n)^\downarrow = \alpha^{-}[n^\downarrow]\) for all \(n \in N\).
Proof. (a) $\Leftrightarrow$ (b). On account of Corollary 7.1, we may assume that $\mathbb{K}$ and $\mathbb{L}$ are the base contexts of two doubly based lattices $\mathbb{K}$ and $\mathbb{L}$, so that $(\alpha, \beta)$ is concept continuous (hence residuated) iff $\alpha$ is induced by the lower adjoint and $\beta$ by the upper adjoint of a common base-preserving complete homomorphism $\varphi : \mathbb{L} \to \mathbb{K}$. Since the specialization orders are induced by the lattice orders, both the upper adjoint $\alpha^*$ and the lower adjoint $\beta_*$ is induced by $\varphi$. It follows that $\varphi$ is a Clr-morphism. Again by Corollary 7.1, this is equivalent to saying that $(\alpha^*, \beta_*)$ is conceptual, hence residuated.

For (a) $\Leftrightarrow$ (d), use the identities $\alpha^{-}[n]\alpha = \beta^{-}[n]\beta$ and $\alpha^{-}[h]\alpha = \beta^{-}[h]\beta$ characterizing concept continuity.

That (c) and (d) are equivalent is immediate from the equivalences

\[
j \in \alpha^{-}[n] \Leftrightarrow n \in \alpha(j)\alpha \quad \text{and} \quad n \in \beta_*[j] \Leftrightarrow j \in \beta_*(n)\beta
\]

\[
h \in \alpha^{-}[m] \Leftrightarrow m \in \alpha^*(h)\alpha \quad \text{and} \quad m \in \beta^{-}[h] \Leftrightarrow h \in \beta(m)\beta.
\]

From the characterizations (c) and (d) it is obvious that the category of posets with residuated and residual maps is embedded in $\mathbb{C}^r$ (by sending $\varphi$ to $(\varphi, \varphi)$), and the category of posets with doubly residuated or residual maps is embedded in $\mathbb{C}_r$ (by sending $\psi$ to $(\psi, \psi^*)$ or $(\psi_*, \psi)$, respectively).

Let us note a few additional properties of residuated mapping pairs.

**Corollary 8.3** If $(\alpha, \beta)$ is a residuated pair then $(\alpha^*, \beta_*)$ is a residual pair, and the following identities are fulfilled:

\[
\alpha^*[-A\alpha] = \beta[A\alpha], \quad \beta_*[-B\beta] = \alpha[B\alpha],
\]

\[
\alpha^*[C\alpha] = \alpha^{-}[C\alpha] = \beta_*[C\alpha], \quad \beta_*[C\alpha] = \beta^{-}[C\alpha],
\]

\[
\alpha^*[D\alpha] = \alpha^{-}[D\alpha] = \beta_*[D\alpha], \quad \beta_*[D\alpha] = \beta^{-}[D\alpha].
\]

**Proof.** For the first equation, observe the equivalences

\[
h \in \alpha^*[-A\alpha] \Leftrightarrow A\alpha \subseteq \alpha^*(h)\alpha \Leftrightarrow \beta[A\alpha] \subseteq h \alpha \Leftrightarrow h \in \beta[A\alpha].
\]

The identity $\alpha^*[C\alpha] = \beta_*[C\alpha]$ follows from conceptuality of $(\alpha^*, \beta_*)$ (see Proposition 3.1), and the identity $\alpha^{-}[C\alpha] = \beta^{-}[C\alpha]$ from concept continuity of $(\alpha, \beta)$ (see Proposition 3.2). For $\alpha^{-}[C\alpha] = \beta_*[C\alpha]$, use the equivalences

\[
j \in \alpha^{-}[C\alpha] \Leftrightarrow C\alpha \subseteq \alpha(j)\alpha \Leftrightarrow \beta_*[C\alpha] \subseteq j \alpha \Leftrightarrow j \in \beta_*[C\alpha].
\]

The other equations are derived analogously. 

Form these remarks, one easily deduces:

**Corollary 8.4** By passing to adjoints, the context categories $\mathbb{C}_r$ and $\mathbb{C}^r$ are dually isomorphic to each other.

Now, we are in a position to establish the main result of this section:
Theorem 8.1  Assigning to each residual pair \((\alpha, \beta)\) the complete homomorphism \(\alpha^{-} = \beta^{-}\), one obtains an equivalence between the category \(C^r\) of purified contexts and the category \(Cl^r\) of doubly based lattices. In the opposite direction, the equivalence functor sends any \(Cl^r\)-morphism \(\varphi\) to the mapping pair built by the restrictions of \(\varphi\) to the join- and meet-bases.

Similarly, associating with any residuated pair \((\alpha, \beta)\) the complete homomorphism \(\alpha^{-} = \beta^{-}\), one obtains a dual equivalence between the categories \(C^r\) and \(Cl^r\). In the opposite direction, the dual equivalence functor sends any \(Cl^r\)-morphism \(\varphi\) to the pair constituted by the restriction of \(\varphi^*\) to the join-bases and the restriction of \(\varphi^*\) to the meet-bases.

Proof. We already know that for any context \(K = (J, M, I)\), the triple \(B^rK = B^r_JK = B^r_MK = (B_K, J_0, M_0)\) is a doubly based lattice, and that
\[
\eta_K = (\gamma_K, \mu_K) : K \rightarrow K^oB^oK, \ x \mapsto (J \cap \downarrow x, M \cap \uparrow x)
\]
is a natural isomorphism provided \(K\) is purified. On the other hand, for an arbitrary doubly based lattice \(K = (K, J, M)\), we have the purified context \(K^oK = K^o_JK = K^o_MK = (J, M, \leq)\) and the natural isomorphism
\[
\iota_K : K \rightarrow B^oK^oK, \ x \mapsto (J \cap \downarrow x, M \cap \uparrow x).
\]
A few technical verifications show that for any \(C^r\)-morphism \((\alpha, \beta)\) the map \(B^r(\alpha, \beta) = \alpha^{-} = \beta^{-}\) has the required properties of a \(Cl^r\)-morphism, and similarly, for any \(C^r\)-morphism \((\alpha, \beta)\), the map \(B_r(\alpha, \beta) = \alpha^{-} = \beta^{-}\) is a \(Cl^r\)-morphism, too.

Conversely, given any \(Cl^r\)-morphism \(\varphi : L \rightarrow K\) between doubly based lattices \(L = (L, H, N)\) and \(K = (K, J, M)\), we have that the restricted maps \(\alpha = \varphi_* : J \rightarrow H\) and \(\beta = \varphi^* : M \rightarrow N\) form a residuated pair \((\alpha, \beta)\), hence a \(C^r\)-morphism. This and the remaining statements are easy consequences of earlier results.

Let us finally put together all pieces of the Galois duality puzzle. In the diagram on the next page, we place 13 different categories in three triangular levels; all categories of one level are mutually equivalent or dual. Each double line symbolizes a categorical equivalence, while each (non-dotted) single line stands for a duality. In the table of morphisms,
\[
\begin{align*}
\vdash J & \text{ indicates that join-bases are preserved,} \\
\vdash M & \text{ indicates that meet-bases are preserved,} \\
\varphi_* & \text{ denotes the lower adjoint and } \varphi^* \text{ the upper adjoint of } \varphi.
\end{align*}
\]
Equivalent and dual categories of contexts and complete lattices

<table>
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9 Reduced Contexts and Small-Based Lattices

We have seen that join- and meet-bases play a crucial role in the passage between context and complete lattices. The situation is simplified considerably if we focus on small-based lattices (cf. [2]); these have a least join-base $J(L)$, consisting of all join-irreducibles, and a least meet-base $M(L)$, consisting of all meet-irreducibles. Of course, all finite lattices have that property. Any small-based lattice is isomorphic to the concept lattice of an up to isomorphism unique reduced context, viz. the standard context or small context $SL = (J(L), M(L), \leq)$. An arbitrary context $K = (J, M, I)$ is reduced iff it is purified, each object concept $\gamma(j)$ is join-irreducible, and each attribute concept $\mu(m)$ is meet-irreducible in the concept lattice $BK$ – in other words, iff $\eta_K$ induces an isomorphism between the contexts $K$ and $SBK$. On the other hand, a complete lattice is small-based iff it is isomorphic to the concept lattice of its standard context. We may regard $S$ as a covariant functor, sending each join- and meet-irreducibility preserving complete homomorphism to the pair of its restrictions to the least join- and meet-bases, respectively. But, of course, there is also a contravariant standard context functor, sending any complete homomorphism, whose lower adjoint preserves join-irreducibility and whose upper adjoint preserves meet-irreducibility, to the pair of those adjoints restricted to the respective least bases. Now, the equivalences and dualities between categories of purified contexts and doubly based lattices derived in the previous sections immediately lead to the following more constrained but technically simpler results:

**Theorem 9.1** Under the covariant concept lattice functor and the standard context functor in the reverse direction, the category of reduced contexts and conceptual morphisms is equivalent to the category of irreducibly bigenerated lattices and complete homomorphisms preserving the least join- and meet-bases. Hence, the conceptual morphisms between reduced contexts are in one-to-one correspondence with the irreducibility preserving complete homomorphisms between their concept lattices.

**Theorem 9.2** Via the contravariant concept lattice functor and the contravariant standard context functor, the category of reduced contexts with concept continuous pairs is dual to the category of irreducibly bigenerated lattices and complete homomorphisms whose lower adjoint preserves the join-bases and whose upper adjoint preserves the meet-bases.

Similarly, the category of reduced contexts with residual (respectively, residuated) mapping pairs is equivalent (respectively, dual) to the category of small-based lattices with complete homomorphisms preserving and reflecting the least bases.
References


