Abstract. In algebra, topology and order theory, decompositions into prime elements or substructures play a crucial role. Often, the mathematical objects in question carry an order structure such that each element \( x \) has a \( \lor \)-pseudocomplement, that is, a least element whose supremum with \( x \) gives the top element. In such posets \( P \), one finds that the essential primes, i.e. those join-prime elements whose \( \lor \)-pseudocomplement is not the least element, are just the atoms of the skeleton \( P^\ast \) (the Boolean poset of all pseudocomplements). A convenient necessary and sufficient condition for the existence of a (unique) irredundant prime decomposition of the top element 1 is that \( P \setminus \{1\} \) has a cofinal subset not containing any binary forks (a specific kind of binary trees). In that case, the Dedekind-MacNeille completion of \( P^\ast \) is isomorphic to the powerset of the irredundant prime decomposition. The exclusion of infinite upper antichains even guarantees a least finite prime decomposition. Our results also provide some interesting consequences concerning the cellularity of ordered sets and topological spaces. As expected, relative pseudocomplements entail still stronger decomposition properties. For example, the existence of at least one essential prime for each non-zero element in a Brouwerian \( \lor \)-semilattice already guarantees unique irredundant prime decompositions for all elements.

0. Introduction

One of the most prominent links between order and topology was made popular by Alexandroff in the thirties, when he pointed out the one-to-one correspondence between quasi-ordered sets, i.e. sets with a reflexive and transitive relation, and Alexandroff spaces, i.e. topological spaces whose topology is closed under arbitrary intersections (Alexandroff called them discrete spaces; see [1]). Today, one formulates that correspondence between ordered and topological structures by saying that the category of Alexandroff spaces (with continuous maps) is concretely isomorphic to the category of quasi-ordered

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sets (with order-preserving = isotone maps). The isomorphism functor associates with any quasi-ordered set $Q = (X, \leq)$ the space $AQ$, whose open sets are the upper sets or up-sets

$$U = \up U = \{ x \in X : u \leq x \text{ for some } u \in U \},$$

while the lattice of closed sets, also referred to as the Alexandroff completion of $Q$ and denoted by $AQ$, consists of all lower sets or down-sets

$$D = \down D = \{ x \in X : x \leq a \text{ for some } a \in D \}.$$

Frequently it is helpful to proceed along the following ‘recycling scheme’:

\begin{center}
\begin{tikzcd}
\text{quasi-ordered sets} & \text{partially ordered sets} \\
\downarrow & \downarrow \\
\text{complete lattices} & \text{closure systems} \\
\downarrow & \downarrow \\
\text{algebraic lattices} & \text{algebraic closure systems} \\
\downarrow & \downarrow \\
\text{spatial coframes} & \text{topological closure systems} \\
\downarrow & \downarrow \\
\text{algebraic coframes} & \text{Alexandroff topologies}
\end{tikzcd}
\end{center}

An algebraic or compactly generated lattice is a complete lattice in which every element is a join of compact elements [2, 6], while a spatial coframe is a complete lattice in which every element is a join of $\lor$-prime elements (see, e.g., [24] for the dual notion). It is well known that, up to isomorphism,

- the complete lattices are the closure systems,
- the algebraic lattices are the algebraic closure systems,
- the spatial coframes are the topological closure systems,
- and the algebraic coframes are the Alexandroff topologies.

One of the earliest references to that theme is Büchi [4]. Recall that a frame or locale is a complete lattice enjoying the distributive law

$$x \land \bigvee Y = \bigvee \{ x \land y : y \in Y \}$$

for arbitrary elements $x$ and subsets $Y$, and a coframe satisfies the dual distributive law. Algebraic spatial coframes are the same thing as superalgebraic lattices, characterized by the property that they have a join-dense set of completely join-prime ($\lor$-prime) elements, or dually, a meet-dense set of completely meet-prime ($\land$-prime) elements [12]. The Axiom of Choice (AC) allows to omit the word ‘spatial’ in the above characterization, on account of
the Birkhoff-Frink Theorem about meet-decompositions in algebraic lattices [2]. Thus, assuming AC, the class of algebraic coframes is closed under dualization; in other words, an algebraic coframe is also a coalgebraic frame, and conversely.

The big jump from partially ordered sets to algebraic coframes may be divided into several intermediate steps:

- \( \lor \)-pseudocomplemented posets
- sectionally \( \lor \)-pseudocomplemented posets
- relatively \( \lor \)-pseudocomplemented \( \lor \)-semilattices
- dual Brouwerian lattices = dual Heyting algebras
- coframes
- spatial coframes
- algebraic coframes

In the present study, we shall be concerned mainly with the most comprehensive of these special classes of ordered sets, namely that of \( \lor \)-pseudocomplemented posets (see below), because it includes all of the other important structures such as (dual) Heyting algebras, in particular Boolean algebras (with applications to mathematical logic, set theory and topology), and their complete version, the coframes. The class of spatial coframes in turn comprises all closed set lattices of topological spaces, but also the dual lattices of radical ideals in rings and other ideal structures, as well as down-set lattices of arbitrary ordered sets. Of course, there are also interesting classes of \( \lor \)-pseudocomplemented complete lattices that fail to be coframes, for example, the convex geometries.

By a \( \lor \)-pseudocomplemented poset we mean a poset that contains a top element 1 and for each element \( x \) a \( \lor \)-pseudocomplement, that is, a least element \( y \) such that the only upper bound of \( x \) and \( y \) is 1, written \( x \lor y = 1 \) (see [22, 25, 33] for the dual notion). In a sectionally \( \lor \)-pseudocomplemented poset, each principal ideal \( \downarrow a = \{ x \in P : x \leq a \} \) is \( \lor \)-pseudocomplemented. Examples of \( \lor \)-pseudocomplemented but not sectionally \( \lor \)-pseudocomplemented complete lattices are easily obtained by adding a new top element to any complete lattice that is not \( \lor \)-pseudocomplemented.

A \( \lor \)-semilattice \( S \) is Brouwerian or relatively pseudocomplemented if for all \( x, y \in S \) there is a relative pseudocomplement \( x \setminus y \) such that

\[
x \leq y \lor z \iff x \setminus y \leq z.
\]

Obviously, such semilattices are sectionally \( \lor \)-pseudocomplemented, but not conversely: in fact, the finite Brouwerian semilattices are just the finite distributive lattices, while many finite lattices like the pentagon are sectionally \( \lor \)-pseudocomplemented but not distributive. The simplest example of a Brouwerian \( \lor \)-semilattice that fails to be a lattice (hence a dual Heyting algebra) is
obtained by forming the ordinal sum of the chain $\omega$ of natural numbers and a three-element join-semilattice that is not a chain. Non-complete bounded chains are Brouwerian lattices but clearly not coframes. Non-atomic complete Boolean lattices like that of all regular open subsets of the reals are examples of non-spatial (co)frames. Finally, most of the spatial coframes of closed sets occurring in classical topology fail to be Alexandroff topologies, hence cannot be algebraic.

In all, we see that we actually have a sequence of proper specializations, starting with $\lor$-pseudocomplemented posets, and ending with algebraic coframes, concretely represented by Alexandroff topologies.

Our approach via joins is in accordance with the work by E. T. Schmidt [31, 32] and is the adequate one for the purposes of a universal join decomposition theory, opposite to the algebraic meet decomposition theory (see Crawley and Dilworth [6]). Join decompositions are the more intuitive ones, because the elements or objects are ‘decomposed into smaller parts’ (note that also in [6], direct decompositions deal with joins rather than meets). Nevertheless, it should be mentioned that the majority of authors prefer the dual notions like ($\land$-)pseudocomplements etc. (see e.g. Birkhoff [2], Frink [18], Grätzer [21], Katriňák [25, 26]).

The material of the present paper is organized as follows.

Section 1 contains the crucial definition of $\kappa$-prime elements and a brief review of recent results about prime decompositions in arbitrary posets (see [14] for more background).

In Section 2, we recall some of the basic definitions and results concerning $\lor$-pseudocomplemented posets. In particular, we give a short proof for the known fact that the skeleton (which consists of all $\lor$-pseudocomplements) has a Boolean normal completion and is, therefore, a Boolean poset. Moreover, we demonstrate that if a $\lor$-pseudocomplemented poset $P$ is meet-dense in a complete lattice having the same top element as $P$, then that lattice is $\lor$-pseudocomplemented, too, and its skeleton is the normal completion of the skeleton of $P$ (Theorem 2).

In Section 3 we show that for any maximal upper antichain (no two elements of which have a common upper bound in the entire poset), the set of its $\lor$-pseudocomplements yields an irredundant decomposition of the top element. This has some interesting consequences for the cellularity of a poset or space.

In Section 4, we provide a whole series of equivalent characterizations for essential primes in a $\lor$-pseudocomplemented poset $P$ by means of the skeleton $P_*$ (Theorem 4); here, an element of $P$ is essential iff its $\lor$-pseudocomplement is not the least element.

Then, in Section 5, we turn to the decomposition theory for $\lor$-pseudocomplemented posets. We show that the top element 1 of $P$ has an irredundant
Prime decomposition iff the remainder \( P−1 \) has a cofinal subset \( Q \) of ‘spoonhandles’, i.e. elements \( q \) for which the principal dual ideal \( \uparrow q = \{ x \in Q : q \leq x \} \) is directed (Theorem 5). More precisely, any irredundant prime decomposition arises as the set of all \( \lor \)-pseudocomplements of any upper antichain of spoon handles, and it consists of all atoms of the skeleton (Theorem 5). As a by-product, we find that the cellularity of \( P \) (the supremum of all cardinalities of upper antichains in \( P−1 \)) is attained by any maximal upper antichain of spoon handles in (a cofinal subset of) \( P−1 \) and coincides with the cardinality of the unique irredundant prime decomposition of the top element. Moreover, we show that 1 has a (least) finite prime decomposition iff \( P−1 \) contains no infinite upper antichains (Theorem 2). This result includes a famous theorem due to Erdős and Tarski [8] about the cellularity. Analogous results hold for \( \kappa \)-primes (Theorem 7).

Finally, in Section 6, we have a closer look at Brouwerian semilattices. It follows from the above results (by relativization to principal ideals) that a Brouwerian \( \lor \)-semilattice is free over the poset of its primes iff no truncated principal ideal contains an infinite upper antichain, which is certainly fulfilled if the whole semilattice does not contain any infinite antichain. In dual Heyting algebras, it suffices to require a meet-dense subset without infinite antichains in order to assure least finite prime decompositions for all elements. This generalizes the known decomposition of ordered sets with no infinite antichains into a finite number of ideals (see Bonnet [3], Diestel [7]).

1. Prime Decompositions and Distributivity in Posets

In order to have the necessary tools at hand, we start with a few general definitions and remarks about prime elements and decompositions in arbitrary posets. For a more systematic treatment of that topic, we refer to [14].

Let \( \kappa \) be any cardinal number or the symbol \( \infty \), and write \( A \subset_\kappa B \) if \( A \) is a subset of \( B \) with fewer than \( \kappa \) elements (where \( A \subset_\infty B \) simply means that \( A \) is a subset of \( B \), i.e. \( A \subseteq B \)). The \( \kappa \)-directed subsets \( D \) of a poset or quasi-ordered set \( P \) are characterized by the property that each subset of \( D \) with fewer than \( \kappa \) elements has an upper bound in \( D \). For \( 2 < \kappa \leq \omega \), it is clear that ‘\( \kappa \)-directed’ means ‘directed’, and an \( \infty \)-directed set is one with a greatest element. An element \( p \in P \) is said to be \( \kappa \)-prime if the set \( \{ x \in P : p \nleq x \} \) is \( \kappa \)-directed. In a \( \lor \)-semilattice, the \( \kappa \)-primes with \( 2 < \kappa \leq \omega \) are the join-primes (\( \lor \)-primes), and in a complete lattice, the \( \infty \)-primes are the completely join-primes (\( \lor \)-primes) in the usual sense.

A join-decomposition of an element \( x \) of a poset \( P \) is a subset with join \( x \), and a prime decomposition is a join-decomposition that consists of \( \omega \)-primes. If \( A \) is a set and \( a \) is an element of \( A \), it will be convenient to write \( A−a \) for \( A \setminus \{ a \} \). A subset \( A \) of a poset \( P \) is called irredundant if for each \( a \in A \), there is an upper bound \( b \) of \( A−a \) with \( a \nleq b \); the latter simply means \( a \nleq \lor(A−a) \).
if that join exists (cf. [6, Ch.7]). Thus, a join-decomposition is irredundant iff it is minimal with respect to inclusion. Every irredundant set is an antichain (a set of pairwise incomparable elements), but not conversely. Stronger than irredundance is the following property: a subset $A$ of $P$ is independent if for each $a$ in $A$, there is an upper bound $b$ of $A - a$ such that $a$ and $b$ are disjoint, and $a$ is not the least element of $P$. In complete lattices, independence means $a \wedge \bigvee (A - a) = 0 \neq a$ for all $a \in A$. An independent join-decomposition is also referred to as a direct decomposition (cf. [6, Ch.8]). Every independent set is irredundant and has pairwise disjoint elements (where disjointness in posets means that there is no common lower bound, except a possible least element). Conversely, in a frame, a set of pairwise disjoint non-zero elements is already independent. (Caution: In the theory of Boolean algebras, ‘irredundance’, ‘independence’ and ‘antichains’ sometimes have a different meaning; see e.g. [27].)

We call an element $p \in P$ $\lor$-essential for an element $x \in P$ if there exists a $q < x$ with $x = p \lor q$. Slightly weaker is the following property: an element $p$ is essential for $x$ if there is a $q$ with $x \not\leq q$ but $\uparrow p \cap \uparrow q \subseteq \uparrow x$ (which means $x \leq p \lor q$ if that join exists). In case $x$ is the greatest element, both conditions are equivalent, and we simply speak of an essential element (see [14]; cf. [7] for the more specific notion of essential ideals, and [30] for the dual notion of essential primes in frames). Any irredundant join-decomposition of $x$ consists of essential elements for $x$. Essential primes are maximal among all primes, but not conversely. However, if an element $x$ has a finite prime decomposition then the essential primes for $x$ are precisely the maximal primes below $x$. The following strong uniqueness theorem for irredundant prime decompositions has been established in [14]:

**Theorem 1.** Let $R$ be a subset of a poset $P$ and call $R$-prime those members of $R$ which are $\omega$-prime in $P$. Then the following are equivalent for any subset $A$ of $P$ and an element $x$ having at least one join-decomposition into $R$-primes:

(a) $A$ is the set of all essential $R$-primes for $x$ and has join $x$.
(b) $A$ is the least join-decomposition of $x$ into maximal $R$-primes in $\downarrow x$.
(c) $A$ is an irredundant join-decomposition of $x$ into $R$-primes.
(d) $A$ consists of maximal $R$-primes and is maximal irredundant in $\downarrow x$.

If $x$ has a finite join-decomposition into $R$-primes then these conditions are also equivalent to each the following statements:

(a $\omega$) $A$ is the set of all maximal $R$-primes in $\downarrow x$ and has join $x$.
(b $\omega$) $A$ is the least finite join-decomposition of $x$ into $R$-primes.
(c $\omega$) $A$ is a finite antichain of $R$-primes with join $x$.
(d $\omega$) $A$ is a maximal irredundant set of maximal $R$-primes in $\downarrow x$. 
Note the subtle difference between (d) and \((d_\omega)\): the former condition includes that \(A\) is maximal among all irredundant sets consisting of elements below \(x\), while in the latter, it is only required that \(A\) be maximal among the irredundant sets of maximal \(R\)-primes below \(x\).

Analogous results hold for \(\kappa\)-prime decompositions (see [14] for details).

In the late seventies, we have initiated a theory of distributive laws for ordered sets (see [9, 10, 11, 12]); later on, the theory was extended from ordered sets to closure spaces, contexts and their concept lattices (see [13, 16]). Independently, similar results about modular, distributive and Boolean ordered sets were obtained by Chajda, Larmerová, Rachůnek [5, 28] and Niederle [29].

Let us briefly recall some of the main ideas and concepts in that area.

As is well known, any poset \(P\) is join- and meet-densely embedded in an up to isomorphism unique complete lattice, its completion by cuts, Dedekind-MacNeille completion or normal completion \(\mathcal{N}P\) (see, for example, [2] and [10, 11]). Concretely, \(\mathcal{N}P\) may be regarded as the closure system of all cuts. The corresponding closure operator \(\Delta\) sends each subset \(A\) to the intersection of all principal ideals containing \(A\). Denoting by \(A^{\downarrow}\) and \(A^{\uparrow}\) the set of all lower and upper bounds of \(A\), respectively, we have \(\Delta A = A^{\uparrow\downarrow}\). The embedding of \(P\) in \(\mathcal{N}P\) associates with each \(x \in P\) the principal ideal \(\downarrow x = \{x\}\). However, it is more convenient to consider \(P\) as a subposet of \(\mathcal{N}P\). The lattice extension (in [29]: characteristic lattice) of \(P\) is then the lattice \(\mathcal{L}P\) generated by \(P\) in \(\mathcal{N}P\). The following basic result characterizes so-called (weakly) distributive ordered sets by various alternative conditions and shows that distributivity is a self-dual property, also in the general setting of posets.

**Proposition 1.** For any poset \(P\), the following are equivalent:

(a) \(\downarrow x \cap \Delta \{y, z\} = \Delta(\downarrow x \cap \downarrow \{y, z\})\) for all \(x, y, z \in P\).

(b) \(\downarrow x \cap \Delta Y = \Delta(\downarrow x \cap \downarrow Y)\) for all \(x \in P\) and all finite \(Y \subseteq P\).

(c) For finite \(Y \subseteq P\) and \(x \in \Delta Y\), there is a \(Z \subseteq \downarrow Y\) with \(x = \bigvee Z\).

(d) \(\Delta\) preserves finite intersections of finitely generated down-sets.

(e) The lattice extension \(\mathcal{L}P\) is distributive.

(f) In \(\mathcal{N}P\), the identity \(x \land (y \lor z) = (x \land y) \lor (x \land z)\) holds for \(x, y, z \in P\).

(g) For all \(x, y, z \in P\), \(\downarrow x \cap \downarrow y \subseteq \downarrow z\) and \(\uparrow x \cap \uparrow z \subseteq \uparrow y\) imply \(y \leq z\).

Stronger are the following mutually equivalent conditions:

(h) \(P\) is strongly distributive, i.e. \(\downarrow x \cap \Delta Y = \Delta(\downarrow x \cap \downarrow Y)\) for all \(Y \subseteq P\).

(i) For all \(Y \subseteq P\) and \(x \in \Delta Y\), there is a \(Z \subseteq \downarrow Y\) with \(x = \bigvee Z\).

(j) \(\Delta\) preserves finite intersections of down-sets.

(k) The normal completion \(\mathcal{N}P\) is a frame.

See [10] and [13] for these and related facts concerning posets with distributive completions by cuts or ideals, respectively.
Note that every \( \kappa \)-prime element \( p \) is \( \kappa \)-irreducible (i.e. \( A \subseteq \kappa, P \) and \( p = \bigvee A \) imply \( p \in A \)), and that the converse holds in strongly distributive posets (in fact, for \( \kappa \leq \omega \), weak distributivity suffices).

2. Pseudocomplemented and Boolean Posets

Two elements \( a \) and \( a' \) of \( P \) are called disjoint (respectively, cojoint) if \( \downarrow a \cap \downarrow a' = \top \) (respectively, \( \uparrow a \cap \uparrow a' = \bot \)). If \( a \) and \( a' \) are both disjoint and cojoint then they are complements of each other. A poset \( P \) is complemented if each of its elements has at least one complement. Distributive complemented posets are called Boolean (cf. [29]). Furthermore, a poset \( P \) is \((\land\neg)\)-pseudocomplemented (cf. [22, 25, 26, 29, 33]) if for each \( a \in P \) there exists a greatest element \( a^* \) disjoint from \( a \), called the \((\land\neg)\)-pseudocomplement of \( a \). The \((\lor\neg)\)-pseudocomplement \( a_* \) and \((\lor\neg)\)-pseudocomplemented posets are defined dually. Usually one postulates a least element \( 0 \) and a greatest element \( 1 \) for (pseudo-)complemented posets; in that case, \( a \) and \( a' \) are disjoint (cojoint) iff \( a \land a' = 0 \) (\( a \lor a' = 1 \)). It is convenient to write \( A^* \) for \( \{ a^* : a \in A \} \) and \( A_* \) for \( \{ a_* : a \in A \} \) (\( A \subseteq P \)). The poset \( P^* \) (respectively, \( P_* \)) is called the \((\lor\neg)\)-Booleanization or the \((\lor\neg)\)-skeleton of \( P \) (see [2, 18, 25, 26, 29]).

The famous Glivenko-Frink Theorem [18], stating that the skeleton of a pseudocomplemented semilattice \( S \) is always Boolean, extends to the poset setting, as demonstrated by Halaš [22, 23] and Niederle [29]. Almost 30 years earlier, Katriňák [25] had already shown that the skeleton of a pseudocomplemented poset is a Boolean lattice iff it is a \( \lor \) or \( \land \) semilattice. We add here a short alternative proof of a slightly stronger result.

**Proposition 2.** If \( P \) is a pseudocomplemented poset then the skeleton \( P^* \) is strongly distributive and uniquely complemented, in particular Boolean.

**Hence, the following assertions are equivalent:**

(a) \( P \) is Boolean.

(b) \( P \) is uniquely complemented.

(c) \( P \) coincides with its \((\lor\neg)\)-skeleton.

**Proof.** The operators \( \Delta \) and \( \downarrow \) refer to \( P^* \). Suppose \( x \in \Delta Y \) for some \( Y \subseteq P^* \). Given \( \downarrow x \cap \downarrow Y \subseteq \downarrow z^* \), we have to show that \( x \leq z^* \) (then \( Z = \downarrow x \cap \downarrow Y \) satisfies \( x = \bigvee_{P^*} Z \)). If \( b \) is a lower bound of \( \{ x, z \} \) in \( P^* \) and \( y \) is an element of \( Y \), then \( b \) and \( y \) are disjoint, because \( \downarrow b \cap \downarrow y \subseteq \downarrow z \cap \downarrow x \cap \downarrow Y \subseteq \downarrow z \cap \downarrow z^* \). Hence, we have \( y \leq b^* \) for all \( y \in Y \), and the hypothesis \( x \in \Delta Y \) yields \( b \leq x \leq b^* \) and so \( b = 0 \). Thus, \( x \) and \( z \) are disjoint, and in fact \( x \leq z^* \).

For \( x \in P^* \), the pseudocomplement \( x^* \) is easily seen to be the unique complement in \( P^* \) (see [22] and the remarks below). Since the map \( x \mapsto x^* \) is a dual automorphism of \( P^* \), it is clear that \( P^* \) must be a lattice whenever it is a \( \lor \) or \( \land \) semilattice.
For the equivalence of (a), (b) and (c), see also Niederle [29].

In view of the intended applications to join-decompositions, from now on we adopt the general assumption that

$P$ is a $\vee$-pseudocomplemented poset and $P_s$ its $\vee$-skeleton.

Let us point out some properties of such posets. $P$ has a top element 1 iff it has a bottom element 0, in which case $0 = 1_s$, and 1 is the only element $a$ with $a_s = 0$. We shall make frequent use of the basic equivalences

$$a \vee b = 1 \text{ in } P \iff a_s \leq b \iff b_s \leq a \iff b_s \leq a_{ss} \iff a_s \land b_s = 0 \text{ in } P_s.$$ 

From these equivalences it follows at once that for each $a \in P_s$ the $\vee$-pseudocomplement $a_s$ is the unique complement in $P_s$. The map $a \mapsto a_s$ is a kernel operator (i.e. $a_{ss} \leq b \iff a_{ss} \leq b_{ss}$), and consequently, in the range $P_s = P_{ss}$, joins coincide with those in $P$.

It is known that the normal completion of a Boolean lattice or poset is again Boolean (see [2, Ch.V.11], [23] and [29]), and that the normal completion of a pseudocomplemented poset is again pseudocomplemented (see [13], [16] and [23]). Stronger is the following theorem, parts of which had been obtained, in a dual form using the language of standard completions, in [16]:

**Theorem 2.** If a $\vee$-pseudocomplemented poset $P$ is meet-dense in a complete lattice $L$ whose top element belongs to $P$, then $L$ is $\vee$-pseudocomplemented, too, and $\vee$-pseudocomplements in $P$ coincide with those in $L$. Furthermore,

$L_s \simeq \mathcal{N}(P_s) \simeq (\mathcal{N}P)_s$ 

is a complete Boolean algebra. Hence, if $P$ is Boolean then so is $\mathcal{N}P$.

**Proof.** Meet-density of $P$ entails that joins in $P$ coincide with those in $L$, and that for $c \in L$, the element $c_s = \bigvee_L \{a_s : a \in P, c \leq a\}$ is the $\vee$-pseudocomplement in $L$; indeed, for any $d \in L$,

$$c_s \leq d \iff \forall a \in P \ (c \leq a \Rightarrow a_s \leq d) \iff$$

$$\forall a, b \in P \ (c \leq a, d \leq b \Rightarrow a_s \leq b) \iff$$

$$\forall a, b \in P \ (c \leq a, d \leq b \Rightarrow a \lor b = 1) \iff c \lor d = 1.$$ 

In order to prove meet-density of $P_s$ in $L_s$, choose a $c \in L$ and a set $A \subseteq P$ with $c_s = \bigwedge_L A$. Then $A_{ss} \subseteq P_s$ and $c_s = \bigwedge_L A_{ss}$. But $P_s$ is also join-dense in $L_s$, since $c = \bigwedge B$ with $B \subseteq P$ implies $B_s \subseteq P_s$ and $c_s = \bigwedge_L B_s$: for $d \in L_s$, we have $c_s \leq d \iff d_s \leq c \iff \forall b \in B \ (d_s \leq b) \iff \forall b \in B \ (b_s \leq d)$.

In all, we see that $P_s$ is both join- and meet-dense in $L_s$, so that $L_s$ is (isomorphic to) the normal completion of $P_s$. The isomorphism $(\mathcal{N}P)_s \simeq \mathcal{N}(P_s)$ is now immediate, by considering the special case $L = \mathcal{N}P$. By the dual of Proposition 2, $(\mathcal{N}P)_s$ is Boolean, because $\mathcal{N}P$ is $\vee$-pseudocomplemented by the first part of the proof. Finally, if $P$ is Boolean then $P = P_s$, and therefore $\mathcal{N}P = \mathcal{N}(P_s) = (\mathcal{N}P)_s$ is Boolean, too. \qed
The above considerations also show that if $P$ is $\kappa$-join-complete then so is $P_\prec$, and in particular, that $P_\prec$ is a Boolean lattice if $P$ is a $\lor$-semilattice.

Observe that a finite poset with Boolean normal completion need not be ($\lor$-)pseudocomplemented. Counterexamples were given in [16] and [23].

3. Antichains and Irredundant Decompositions

In Boolean lattices, the $\lor$-primes are just the atoms (that is, the minimal non-zero elements). Surprisingly, much of the decomposition theory for Boolean algebras and for topological spaces extends to the considerably more general setting of pseudocomplemented semilattices or posets (no distributive laws assumed).

The following definition will be crucial for our investigation of join-decompositions: an upper (lower) antichain in a poset is a subset of elements no two of which have a common upper (lower) bound in the entire poset. Upper antichains are also termed strong antichains. Clearly, such sets are antichains, and every set of maximal elements is an upper antichain, but neither of these implications can be inverted. The system of all upper (lower) antichains is of finite character, so by Zorn’s or Tukey’s Lemma, every upper (lower) antichain is contained in a maximal one. Henceforth, suppose that

$P$ is a $\lor$-pseudocomplemented poset with greatest element $1$, and $Q$ is a cofinal subset of $P_\prec$, that is, $\downarrow Q = P_\prec$.

Consequently, a subset $A$ of $P$ has an upper bound in $Q$ iff $1$ is not the join of $A$ in $P$, and $Q_\prec$ is coinitial (i.e. dually cofinal) in $P_\prec - 0$. Note also that the sets of pairwise disjoint (cojoint) elements in $P$ are precisely the lower (upper) antichains in $P_\prec - 0$ ($P_\prec - 1$). Let us call a subset $A$ of $P$ $\ast$-faithful if the map $a \mapsto a_\ast$ from $A$ onto $A_\ast$ is one-to-one. Every upper antichain $A$ in $P_\prec - 1$ is $\ast$-faithful, because for distinct $a, b \in A$, we have $a_\ast \leq b$ but $b_\ast \nleq b$. Moreover, every subset of a Boolean lattice is $\ast$-faithful (since $a \mapsto a_\ast$ is then bijective on the whole lattice). We are now ready for a useful generalization of the known result that in a Boolean lattice, the maximal pairwise disjoint families are the direct decompositions of the top element (cf. [27, Ch.1,3.6]).

**Theorem 3.** Consider the following conditions on a set $A \subseteq Q$:

(a) $A$ is a maximal upper antichain in $Q$.
(b) $A_\ast$ is a maximal lower antichain in $Q_\ast$.
(c) $A_\ast$ is a lower antichain in $Q_\ast$ with $\lor A_\ast = 1$ (in $P$ or in $P_\prec$).
(d) $A_\ast$ is a direct decomposition of $1$ (‘partition of unity’) in $P_\prec$.
(e) $A_\ast$ is an irredundant join-decomposition of $1$ in $P$.

The implications (a) $\Rightarrow$ (b) $\iff$ (c) $\iff$ (d) $\Rightarrow$ (e) are always true.

If $A$ is $\ast$-faithful, the first four conditions (a) – (d) are equivalent.
Proof. (a) $\Rightarrow$ (b). Use the equivalence $a \lor b = 1$ in $P$ $\iff$ $a_\ast \land b_\ast = 0$ in $P_\ast$.

(b) $\Rightarrow$ (d). If $A_\ast$ had an upper bound $b < 1$, we could assume $b \in Q$. Then $a_\ast \leq b$, i.e. $a_\ast \land b_\ast = 0$ in $P_\ast$ for all $a \in A$, whence $b_\ast \in A_\ast$ by maximality of $A_\ast$. But then $b_\ast \leq b = b \lor b_\ast = 1$, a contradiction. The decomposition $1 = \bigvee A_\ast$ in $P_\ast$ is direct, because for each $a \in A$ the element $a_\ast$ satisfies $a_\ast \land a_\ast \ast = 0$ in $P_\ast$ and is an upper bound of $A_\ast - a_\ast$; indeed, $b \in A$ and $b_\ast \neq a_\ast$ imply $a_\ast \land b_\ast = 0$ in $P_\ast$, hence $b_\ast \leq a_\ast$.

(d) $\Rightarrow$ (c). Clear, since $0 \notin Q_\ast$ and joins in $P_\ast$ agree with those in $P$.

(c) $\Rightarrow$ (b). If some $b \in Q$ would satisfy $a_\ast \land b_\ast = 0$ in $P_\ast$ for all $a \in A$, then $a_\ast \leq b$ for all $a \in A$ and $1 = \bigvee A_\ast \leq b$, which is excluded by $Q \subseteq P - 1$.

(d) $\Rightarrow$ (e). If $b < 1$ would be an upper bound of $A_\ast$ in $P$ then $b_\ast \ast$ would be an upper bound of $A_\ast$ in $P_\ast - 1$. Clearly, $A_\ast$ is an irredundant lower antichain in $Q_\ast$, being a direct decomposition.

Under the hypothesis that $A$ is $*$-faithful, we derive (a) from (c).

For distinct $a, b \in A$, we have $a_\ast \neq b_\ast$ and therefore $a_\ast \land b_\ast = 0$ in $P_\ast$, hence $a \lor b = 1$. Thus $A$ is an upper antichain. Concerning maximality, observe that no $b \in Q$ can satisfy $a \lor b = 1$ for all $a \in A$, because the latter would entail $1 = \bigvee A_\ast \leq b$.

It is plain that (b) does not imply (a) in general (consider a chain with more than two elements). Furthermore, even in finite Boolean lattices, condition (e) is strictly weaker than the other ones: for any set $A$ of two atoms, the set $A_\ast$ is an irredundant join-decomposition of the top element (into coatoms), but not a direct decomposition if there are more than two atoms.

Generalizing the notion of cellularity for topological spaces and Boolean algebras (see [27, Ch.1.3]), one defines the (lower) cellularity of a poset $P$ to be the supremum of all cardinalities of subsets of pairwise disjoint elements. The upper cellularity of a poset is the lower cellularity of the dual poset (cf. [27, Ch.2.4], where a slightly different definition of cellularity is given). As an immediate consequence of Theorem 3, we note:

**Proposition 3.** The upper cellularity of $P$ (and of $Q \cup \{1\}$) coincides with the upper and the lower cellularity of the self-dual skeleton $P_\ast$; hence, it is the supremum of all cardinalities of direct decompositions for 1 in $P_\ast$.

The dual of that result is an abstract order-theoretical generalization of the fact that the cellularity of a topology is the same as the cellularity of the complete Boolean algebra of its regular open sets.
An analogous equality holds for the lower (upper) saturation, the least cardinal number greater than all cardinalities of pairwise disjoint (cojoint) subsets (see Erdős and Tarski [8] and [27, Ch.1.3]).

4. Spoons, Forks and Primes

Well-known characterizations of atoms in Boolean algebras extend to the situation of $\lor$-pseudocomplemented posets. Moreover, there is an interesting connection between atoms, primes and so-called ‘spoonhandles’, i.e. elements $q \in Q$ for which the principal dual ideal $\downarrow_Q q = \{ x \in Q : q \leq x \}$ is directed (a ‘spoon’). We prove a more general result on $\kappa$-primes and $\kappa$-spoonhandles, i.e. elements generating a $\kappa$-directed principal dual ideal. By cofinality of $Q$ in $P$, an element $q$ is a $\kappa$-spoonhandle in $Q$ iff 1 is $\kappa$-prime in $\uparrow_P q = \{ x \in P : q \leq x \}$ (for $\kappa > \omega$, this equivalence requires choice).

**Proposition 4.** Suppose $p$ and $r$ are elements of $P$ with $r < 1 = p \lor r$.

1. If $p$ is $\kappa$-prime then $\downarrow_Q r = \{ x \in Q : r \leq x \}$ is $\kappa$-directed.
2. Conversely, if $\downarrow_Q r$ is $\kappa$-directed then $r_\ast$ is $\kappa$-prime.
3. An element $q \in Q$ is a $\kappa$-spoonhandle in $Q$ iff $q_\ast$ is $\kappa$-prime in $P$ (and also in $P_\ast$).

**Proof.** (1) Suppose $A \subseteq_\kappa \downarrow_Q r$. Then each $a \in A$ fulfils $r \leq a \neq 1$, hence $p \not\leq a$ (as $p \lor r = 1$). By $\kappa$-primeness of $p$, we find an upper bound $b$ of $A$ with $p \not\leq b$, in particular $b \neq 1$. This $b$ is dominated by some element of $Q$, which is then an upper bound of $A$ in $\downarrow_Q r$, provided $A$ is nonempty. But if $A = \emptyset$ then any element of $\downarrow_Q r$ (which exists by cofinality of $Q$ in $P-1$) is trivially an upper bound of $A$ in $\downarrow_Q r$.

(2) Assume $\downarrow_Q r$ is $\kappa$-directed, and put $p = r_\ast$. If $A \subseteq_\kappa P$ and $p \not\leq a$ for all $a \in A$ then there is a set $B = \{ b_a : a \in A \} \subseteq_\kappa Q$ such that each $b_a$ is an upper bound of $\{ a, r \}$ (use cofinality of $Q$ in $P-1$ and the Axiom of Choice). It follows that $B \subseteq_\kappa \downarrow_Q r$, whence $B$ has an upper bound $b$ in $Q$, which is then also an upper bound of $A$. It cannot happen that $p \leq b$, because otherwise $1 = p \lor r \leq b$. This shows that $p$ is $\kappa$-prime in $P$.

(3) is a consequence of (1) and (2). If $A \subseteq_\kappa P_\ast$ then $A \subseteq \downarrow b$ implies $A = A_{\ast\ast} \subseteq \downarrow b_{\ast\ast}$. Since $q_\ast \not\leq b$ is equivalent to $q_\ast \not\leq b_{\ast\ast}$, we see that whenever $q_\ast$ is $\kappa$-prime in $P$, it is also $\kappa$-prime in $P_\ast$. $\Box$

Note that an element $p \in P$ is essential (i.e. $p \lor r = 1$ for some $r < 1$) iff $p_\ast$ is not the top element. Clearly, in a Boolean algebra, all $\lor$-primes are atoms and essential. But observe that, in contrast to the case of Boolean algebras, a (finitely) $\lor$-irreducible element of a Boolean poset $P = P_\ast$ need not be (completely) $\lor$-irreducible (meaning that it need not belong to every subset whose join it is). This discrepancy between the poset and the lattice case is witnessed by
Example 1. Take top and bottom, atoms and coatoms of an infinite powerset lattice. This gives a Boolean poset whose coatoms are $\lor$-irreducible but not $\lor$-prime.

Now to the characterizations of atoms in the $\lor$-skeleton $P_\ast$.

**Theorem 4.** For $p \in P$ and $2 < \kappa \leq \infty$, the following are equivalent:

(a) $p$ is an atom of $P_\ast$.

(b) $p$ is $\lor$-prime in $P_\ast$.

(c) $p$ is $\lor$-irreducible in $P_\ast$ (i.e. $p \lor A$ in $P_\ast$ implies $p \in A$).

(d) $p$ is $\lor$-prime in $P_\ast$ (i.e. if $A \subset P_\ast$ has a join above $p$ then $p \in \↓ A$).

(e) $p$ is $\omega$-prime in $P$ and an element of $P_\ast$.

(f) $p$ is an essential $\omega$-prime of $P$.

(g) $p$ is the $\lor$-pseudocomplement of a spoonhandle in $Q$.

(h) $p = q_\ast$ for some $q \in Q$ such that 1 is $\omega$-prime in $\↑ p q$.

**Proof.** (a) $\Rightarrow$ (b). The equivalences $p \not\leq x \iff p \land x = 0$ in $P_\ast \iff x \leq p_\ast$ for $x \in P_\ast$ show that $P_\ast \\setminus \↑ p$ is the principal ideal $\↓ p_\ast = \{ x \in P_\ast : x \leq p_\ast \}$.

The implications (b) $\Rightarrow$ (c) and (b) $\Rightarrow$ (d) are obvious.

(c) $\Rightarrow$ (a). If $p$ is $\lor$-irreducible in $P_\ast$ then also in the normal completion $\mathcal{N}(P_\ast)$, since $P_\ast$ is join-dense in $\mathcal{N}(P_\ast)$. Hence, $p$ is an atom of the Boolean algebra $\mathcal{N}(P_\ast)$ and therefore also an atom of $P_\ast$.

(d) $\Rightarrow$ (a). The assumption $a_\ast < p$ together with $p \leq 1 = a_\ast \lor a_\ast$ leads to $a_\ast < p \leq a_\ast$, hence to $a_\ast = a_\ast \land a_\ast = 0$ in $P_\ast$. Thus, $p$ is an atom of $P_\ast$.

(a) $\Rightarrow$ (e). It suffices to show that $p = q_\ast$ is 3-prime in $P$ if it is an atom in $P_\ast$. Assume $p \not\leq a_i$ ($i = 1, 2$). Then there are $b_i < 1$ with $q \leq b_i$ and $a_i \leq b_i$. It follows that $0 < b_i \leq p$ and consequently $p = b_1 = b_2$. The hypothesis $b_1 < 1$ entails $a_2 \not\leq b_1$ (otherwise $1 = q \lor p = q \lor b_2 \leq q \lor a_2 \leq b_1$). Hence, there is a $c < 1$ such that $a_2 \leq c$ and $a_1 \leq b_1 \leq c$. But $q \leq b_1 \leq c < 1$ together with $p \lor q = 1$ excludes $p \leq c$. Thus $p$ is 3-prime and so $\omega$-prime.

(e) $\Rightarrow$ (f). $p_\ast = 1$ would entail $p = p_\ast = 0$, contradicting $\omega$-primeness.

(f) $\Rightarrow$ (g). If $p$ is an essential $\omega$-prime in $P$ then $p_\ast$ is a spoonhandle in $P_\ast$ 1, by Proposition 4 (1), applied to $\kappa = \omega$, $p_\ast \neq 1$ in place of $r$, and $P_\ast$ instead of $Q$. Since $Q$ is cofinal in $P_\ast$, we have $p_\ast \leq q$ for some $q \in Q$, in particular $p \not\leq q$ (otherwise $1 = p \lor p_\ast = q$). It follows that $q$ is a spoonhandle in $Q$ with $q_\ast = p$. Indeed, $p_\ast \leq q$ yields $q_\ast \leq p$, and $p \not\leq q$ together with $p \leq 1 = q \lor q_\ast$ entails $p \leq q_\ast$, because $p$ is $\lor$-prime.

From Proposition 4 (3), applied to $\kappa = \omega$, we infer the implications (g) $\Rightarrow$ (e) and (e) $\Rightarrow$ (b) with $\kappa = \omega$. Finally, the equivalence of (g) and (h) is clear by cofinality of $Q$ in $P \setminus 1$ and the definition of spoonhandles. \qed
By a *fork* in a poset, we mean a non-linearly ordered subset $Y$ having a least element but no incomparable elements with a common upper bound in the entire poset. (This notion has to be distinguished from that of a *tree*, where upper bounds are excluded only in the tree itself, and linearity is not forbidden).

A *binary fork* is a countable fork $Y = \{ y_n : n \in \mathbb{N} = \omega \setminus \{0\} \}$ such that $y_m < y_n$ iff $2^km \leq n < 2^k(m + 1)$ for some $k \in \mathbb{N}$.

Of course, any such binary fork contains infinite chains, but also infinite upper antichains, for example $\{ y_{2^k + 1} : k \in \mathbb{N} \}$ or $\{ y_{2^k - 2} : k \in \mathbb{N} - 1 \}$.

**Lemma 1.** For an arbitrary poset, the following are equivalent:

(a) There exists a cofinal subset consisting of spoonhandles.

(b) There exists a cofinal subset not containing any fork.

(c) There exists a cofinal subset not containing any binary fork.

Each of these conditions is fulfilled if

(d) there exists a cofinal subset without infinite upper antichains.

**Proof.** The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) and (d) $\Rightarrow$ (c) are obvious. The implication (c) $\Rightarrow$ (a) is obtained by contraposition. If some $\uparrow q$ contains no spoonhandle then in every cofinal subset $C$, one may construct, using dependent choices, a binary fork: choose $y_1 \in \uparrow q \cap C$; then $\uparrow y_1$ is not directed; choose $y_2$ and $y_3$ in $\uparrow y_1 \cap C$ with no common upper bound, and so on (cf. [7]).

We shall call a poset with the above equivalent properties *spoonful*. 
5. Irredundant Prime Decompositions

Now, we are prepared for the main results in this paper. They do not only include necessary and sufficient conditions for the existence of irredundant prime decompositions in lower pseudocomplemented posets, but also provide the ‘essential’ (and necessary) tools in order to find such decompositions.

**Proposition 5.** Let $P$ be a $\lor$-pseudocomplemented poset with top element 1, and assume that $P-1$ has a cofinal subset $Q$ of spoonhandles. Furthermore, let $A$ be a maximal upper antichain $A$ in $Q$. Then:

1. The set $A_*$ is the least, hence unique irredundant prime decomposition of 1 in $P$ and also the direct decomposition into atoms of the skeleton $P_*$. 
2. The upper cellularity of $P$ is attained by $A$ and is therefore equal to the cardinality of the least prime decomposition of 1.
3. The normal completion $\mathcal{N}(P_*) \simeq (\mathcal{N}P)_*$ is a complete atomic Boolean algebra, isomorphic to the powersets of $A$ and of $A_*$. 

**Proof.** (1) By Theorem 4, $A_*$ consists of $\omega$-primes in $P$, and these are atoms of $P_*$. By Theorem 3, $A_*$ is an irredundant join-decomposition of 1 in $P$ and a direct decomposition in $P_*$. By Theorem 1, such decompositions are unique.

(2) If $A'$ is any upper antichain in $P-1$ then, by cofinality of $Q$, there is a (maximal) upper antichain $A$ in $Q$ with $A' \subseteq \downarrow A$, and it follows that each $a' \in A'$ is dominated by a unique $a \in A$. This shows that the cardinality of $A'$ is less than or equal to that of $A$, which in turn coincides with that of $A_*$ (because $A$ is $*$-faithful).

(3) This part is clear by (1) and the obvious fact that the atoms of $P_*$ are precisely those of $\mathcal{N}(P_*)$. \qed

From the previous facts, we now deduce a collection of necessary and sufficient criteria for the existence of irredundant prime decompositions.

**Theorem 5.** For a $\lor$-pseudocomplemented poset $P$ with top element 1, the following are equivalent:

(a) $P-1$ is spoonful.
(b) 1 has a (unique) irredundant prime decomposition in $P$.
(c) 1 has a (unique) irredundant prime decomposition in $P_*$. 
(d) $P_*$ is atomistic (each element is a join of atoms).
(e) $P_*$ is atomic (the set of atoms is coinitial in $P_*-0$).
(f) $P_*$ is coatomic (the set of coatoms is cofinal in $P_*-1$).
(g) $\mathcal{N}(P_*)$ is a complete atomic Boolean algebra.

Sufficient but not necessary for these properties are the following conditions:

(h) $P$ is coatomic, i.e. $P-1$ has a cofinal antichain.
(i) $P-1$ has a cofinal subset with no infinite (ascending) chains.
Proof. For (a) $\Rightarrow$ (b), apply Proposition 5 (1).

(b) $\Rightarrow$ (c). Use the fact that an irredundant prime decomposition is unique and consists of essential $\omega$-primes (Theorem 1), and that these are $\omega$-prime atoms of the $\vee$-skeleton $P_* \ (\text{Theorem 4})$.

(a) $\Rightarrow$ (d). Assume $x_* \not\leq y_*$ in $P_*$. Then there is a $b < 1$ with $x \leq b$ and $y_* \leq b$, and by (a), we may assume that $b$ is a spoonhandle in $P-1$. Hence, by Theorem 4, $b_*$ is an atom of $P_*$ with $b_* \leq x_*$ but $b_* \not\leq y_*$ (otherwise $b = b \vee b_*$).

(e) $\Rightarrow$ (a). For $b \in P-1$, we have $b_* \in P_*-0$, so there is an atom $a \in P_*$ with $a \leq b_*$. By Theorem 4 and Proposition 4, it follows that $a_*$ is a spoonhandle in $P-1$. It cannot happen that $a_* \vee b = 1$ (otherwise $a \leq b_* \leq a_*$ and then $a_* = a \vee a_* = 1, a = a_*$).

In all, we have established the implication circles

(a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (e) $\Rightarrow$ (a) and (a) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (a).

For (e) $\Leftrightarrow$ (f) use the dual automorphism $a \mapsto a_*$ of $P_*$, and for (e) $\Leftrightarrow$ (g) apply Theorem 2 about the normal completion of $P_*$. The implications (h) $\Leftrightarrow$ (i) $\Rightarrow$ (a) and (j) $\Rightarrow$ (k) $\Rightarrow$ (l) $\Rightarrow$ (a) are clear. □

Replacing $P$ with $P_* = P^{**}$, we see that the conditions in Theorem 5 are also equivalent to the postulate that $P_*-1$ be spoonful.

Notice that the exclusion of binary forks is a much stronger property than the existence of a cofinal subset of $P-1$ containing no binary forks.

Example 2. The tree of all finite or infinite binary words (0-1-sequences), ordered by the prefix relation, has a cofinal upper antichain (viz. the set of all infinite words), which clearly does not contain any binary fork. Adding a top element to the tree, one obtains a coatomic and $\vee$-pseudocomplemented complete lattice; for any word with first letter $a$, the singleton word $1-a$ is the $\vee$-pseudocomplement. In that lattice, conditions (h) and (i) are fulfilled, whereas (j), (k) and (l) are violated. On the other hand, the chain $\omega+1$ satisfies the conditions (j), (k) and (l) but neither (h) nor (i).

Invoking Theorem 3 once more, we arrive at
Corollary 1. If the equivalent conditions in Theorem 5 are fulfilled then the upper cellularity of $P$ is, at the same time,
the upper and lower cellularity of $P_*$,
the cardinality of the set of all essential primes in $P$,
the cardinality of the set of all (co)atoms in $P_*$,
the cardinality of the irredundant prime decomposition of 1 in $P$,
the cardinality of any maximal upper antichain of spoonhandles in $P – 1$.

Concerning finite decompositions, the previous considerations yield:

Corollary 2. For a $\lor$-pseudocomplemented poset $P$, the following are equivalent:
(a) $P$ has finite upper cellularity.
(b) $P – 1$ contains no infinite upper antichains.
(c) 1 has a finite prime decomposition.
(d) 1 has a least, hence unique irredundant finite prime decomposition.
(e) The skeleton $P_*$ is finite.
(f) The normal completion $N(P_*)$ is a finite Boolean algebra.
If these conditions are fulfilled then the upper cellularity of $P$ equals the number of maximal (=essential) primes.

In (a) and (b), $P – 1$ may be replaced with a cofinal subset $Q$ (because the upper antichains in $Q$ are also upper antichains in $P – 1$ and any upper antichain in $P – 1$ is dominated by an upper antichain in $Q$).

An old theorem due to Erdős and Tarski [8] states that the lower/upper cellularity of a poset $Q$ is finite if $Q$ does not contain any infinite subset of pairwise disjoint/cojoint elements (in other words, that the saturation never can be $\omega$). This result is an immediate consequence of Theorem 2, by taking for $P$ the down-set coframe of a poset $Q$ and observing that the complements of principal filters form a cofinal subset of $AQ \setminus \{Q\}$ that is isomorphic to $Q$. See also Proposition 3.

For semilattices, some of the previous results may be improved:

Corollary 3. The skeleton $S_*$ of a $\lor$-pseudocomplemented $\lor$-semilattice $S$ is an atomic Boolean algebra iff $S – 1$ is spoonful. Moreover, $S_*$ is a finite Boolean algebra iff $S – 1$ contains no infinite upper antichains.

Immediate consequences are the known facts that a Boolean algebra $B$ is (co)atomic iff $B – 1$ has a cofinal subset containing no binary fork, and that a Boolean algebra with no infinite disjoint subset is already finite.
6. Decompositions in Brouwerian Semilattices

In this final section, we have a look at situations where not only the greatest element but all elements possess an irredundant prime decomposition.

Recall that a relatively \((\lor\lor)-\)pseudocomplemented or Brouwerian \(\lor\)-semi-
lattice is a \(\lor\)-semilattice \(S\) such that for any two elements \(x, y \in S\) there is a relative \(\lor\)-pseudocomplement \(x \setminus y\) satisfying the equivalence
\[x \setminus y \leq z \iff x \leq y \lor z .\]
The complete Brouwerian \(\lor\)-semilattices are exactly the coframes. From the early work of E.T. Schmidt (see [31] or [32]), we know that
every Brouwerian \(\lor\)-semilattice \(S\) is distributive, i.e.
\[x \leq y \lor z \implies x = y_0 \lor z_0 \text{ for suitable } y_0 \leq y \text{ and } z_0 \leq z .\]
Hence, the \(\lor\)-irreducible elements of \(S\) are exactly the \(\lor\)-primes.

Each principal ideal \(\downarrow x\) of a Brouwerian \(\lor\)-semilattice \(S\) is clearly \(\lor\)-pseudo-
complemented (with \(y^* = x \setminus y\) in \(\downarrow x\)); otherwise stated, \(S\) is sectionally \(\lor\)-pseudocomplemented, as mentioned in the introduction. Hence, from Proposition 5 we infer the following necessary and sufficient condition for the existence of irredundant prime decompositions:

**Corollary 4.** Let \(S\) be a Brouwerian \(\lor\)-semilattice. Then, an element \(x\) of \(S\) has an irredundant prime decomposition iff the truncated principal ideal \([0, x]\) is spoonful – in other words, iff for each \(a < x\), there is some \(b \in [a, x]\) such that \(x\) is \(\lor\)-prime in the interval \([b, x]\).

Here, half-open intervals and closed intervals are defined by
\[[a, x] = \{ y \in L : a \leq y < x \}\] and \([b, x] = \{ y \in L : b \leq y \leq x \}\).
Note that \(\lor\)-primes of \([0, x]\) are also \(\lor\)-prime in \(S\), because \(S\) is distributive.

**Corollary 5.** A coframe is spatial (dually isomorphic to a topology) whenever all truncated principal ideals are spoonful.

The latter condition is certainly fulfilled in strongly coatomic coframes, where each element of any truncated principal ideal \([0, x]\) is dominated by a maximal one. But here a sharper result is valid (cf. Gorbunov [20]):

**Corollary 6.** A complete lattice is a strongly coatomic coframe if and only if each of its elements has an irredundant decomposition into (completely) \(\lor\)-irreducible \(\lor\)-primes.

By relativization of Theorem 2, we obtain:

**Corollary 7.** An element \(x\) in a Brouwerian \(\lor\)-semilattice has a (least) finite prime decomposition iff \([0, x]\) contains no infinite upper antichain.

An interesting variant of Corollary 7 holds for dual Heyting algebras, i.e. for Brouwerian \(\lor\)-semilattices that happen to be lattices.
Corollary 8. If a dual Heyting algebra $S$ has a meet-dense subset in which all antichains are finite then $S$ is a free $\lor$-semilattice over the poset $P$ of $\lor$-primes. Thus, each element has a least finite prime decomposition, and $S$ is isomorphic to the system of all finitely generated down-sets of $P$.

Proof. If $M$ is a meet-dense subset containing no infinite antichains, then for each element $x$, the set $M_x = \{ m \land x : m \in M, x \not\leq m \}$ is cofinal in $[0, x[$; indeed, for $a < x$ there is an $m \in M$ with $a \leq m$ but $x \not\leq m$, and it follows that $a \leq m \land x < x$. Clearly, $M_x$ does not contain any infinite antichain, and in particular, $[0, x[$ cannot contain infinite upper antichains. Thus, Corollary 7 applies. For the remaining statements, see [14].

It would be of interest to know whether the converse of the last corollary holds as well. The non-algebraic spatial coframe of all closed sets in the cofinite topology on $\omega$ does have meet-dense subsets without infinite antichains. For instance, one may take the set $\{ n \setminus \{ k \} : k \in n \in \omega \}$.

For algebraic coframes, we can prove the converse of the above result indeed. A poset $P$ is said to be principally separated (see [11], [12]) if for $x \not\leq y$ there exist $p, q$ with $p \leq x, y \leq q, p \not\leq q$, and $\uparrow p \cup \downarrow q = P$. In that case, $p$ is $\omega$-prime, and $q$ is dually $\omega$-prime. For the following characterizations of principally separated posets, see Corollary 4.4. in [12]:

Lemma 2. For a poset $P$, the following are equivalent:

(a) $P$ is principally separated.
(b) Each element has a join-decomposition into $\omega$-primes.
(c) The normal completion of $P$ is superalgebraic.
(d) The normal completion of $P$ is principally separated.

In principally separated posets, there is a close connection between $\omega$-primes and $\omega$-primes:

Lemma 3. An element $p$ of a principally separated poset $P$ is $\omega$-prime iff it is a directed join of $\omega$-primes, or equivalently, the set of all $\omega$-primes below $p$ is directed.

Proof. Let $a$ and $b$ be $\omega$-primes below some $\omega$-prime $p$, and define $a^\lor = \bigvee \{ x \in P : a \not\leq x \}$. Then $p \not\leq a^\lor$ and $p \not\leq b^\lor$, where $P$ is the disjoint union of $\uparrow a$ and $\downarrow a^\lor$, and analogously for $b$. By $\omega$-primeness of $p$, there exists a $c$ with $a^\lor \leq c$ and $b^\lor \leq c$ but $p \not\leq c$. By join-density of the $\omega$-primes, there is an $\omega$-prime $d$ with $d \leq p$ but $d \not\leq c$. It follows that $d \not\leq a^\lor$ and $d \not\leq b^\lor$, hence $a, b \leq d \leq p$. This shows that the set of all $\omega$-primes below $p$ is directed, and by principal separation, $p$ is the join of these. On the other hand, directed joins of $\omega$-primes and, in particular, of $\omega$-primes are $\omega$-prime (see [14]).

Corollary 9. In an algebraic coframe, each element has a finite prime decomposition iff there is no infinite antichain of $\omega$-primes.
Proof. In view of Corollary 8, it remains to show that for an infinite antichain \( A \) of \( \omega \)-primes, the join \( x = \bigvee A \) has no finite prime decomposition. But that is easy with the help of Lemma 3: if \( D \) is a prime decomposition of \( x \) then each \( a \in A \) is dominated by some \( d_a \in D \), and \( a \neq b \) implies \( d_a \neq d_b \). (Indeed, if \( a, b \leq d \in D \) then there is an \( \omega \)-prime \( c \) with \( a, b \leq c \leq d \), and \( c \) in turn lies below some \( a' \in A \), so that \( a = b = c = a' \), because \( A \) is an antichain.) Thus, \( D \) cannot be finite.

Since algebraic coframes are, up to isomorphism, just the down-set lattices of arbitrary posets, Corollary 9 may be regarded as a lattice-theoretical formulation of the fact that a poset has no infinite antichains iff each down-set is a finite union of ideals (see Bonnet [3] and [17, Ch.4.5]).

Generalizing a well-known notion from complete lattices to posets, we call a poset \( P \) lower continuous if for each down-directed subset \( D \) and each \( x \in P \), with \( D \subseteq \downarrow x \), there exists a down-directed \( D' \subseteq \uparrow D \) with \( x = \bigwedge D' \). If \( P \) is down-complete, i.e. each down-directed subset \( D \) has a meet \( \bigwedge D \), then \( D \subseteq \downarrow x \) means \( \bigwedge D \leq x \). The lower continuous complete lattices are just the join-continuous lattices (cf. [19]), characterized by the identity

\[ x \lor \bigwedge D = \bigwedge \{ x \lor d : d \in D \} \]

for all elements \( x \) and all down-directed subsets \( D \); for the dual notion of upper continuous lattices, see [6], and for more general results on so-called \( \mathcal{M} \)-distributive lattices, cf. [10]. In [14], we have established the following existence criterion for irredundant join-decompositions into elements of an arbitrary subset \( R \) (see Chapter 6 of [6] for the restricted dual case of algebraic lattices):

**Lemma 4.** Let \( R \) be a subset of a lower continuous down-complete poset \( P \). If for each non-zero element of \( P \) there is an essential element in \( R \), then each element has an irredundant decomposition into elements of \( R \).

While the proof of that result leans on the down-completeness assumption, there exists a ‘non-complete’ variant for Brouwerian \( \lor \)-semilattices. Note that the coframes, i.e. the complete Brouwerian \( \lor \)-semilattices are just the lower continuous and distributive complete lattices. The \( \omega \)-primes that belong to a distinguished subset \( R \) have been referred to as \( R \)-primes.

**Theorem 6.** If for each non-minimal element of a Brouwerian \( \lor \)-semilattice there is an essential \( R \)-prime, then each element has a (unique) irredundant join-decomposition into \( R \)-primes, and conversely.

**Proof.** Let \( A \) be the set of all essential \( R \)-primes for \( x \) (see Section 1). If \( x \) would not be the join of \( A \), we could choose a \( y \) with \( A \subseteq \downarrow y \) but \( x \not\leq y \). The relative \( \lor \)-pseudocomplement \( x \setminus y \) is the least element \( z \) whose join with \( y \) dominates \( x \). In particular, \( z = x \setminus y \) satisfies \( x \leq y \lor z \) and \( z \neq 0 \). Hence, we find an essential \( R \)-prime \( p \) for \( z \) and a \( q \) with \( z \not\leq q \) but \( z \leq p \lor q \). By
definition of $z$, the join $y \lor q$ cannot be above $x$, whereas $x \leq y \lor z \leq p \lor y \lor q$, showing that $p$ is also essential for $x$. Thus, $p$ belongs to $A$, whence $p \leq y$. But then $x \leq p \lor y \lor q = y \lor q$, a contradiction. By way of contraposition, $x$ must be the join of the set $A$, and by Theorem 1, $A$ is the unique irredundant join-decomposition of $x$ into $R$-primes.

The converse implication follows from the fact that elements of an irredundant join-decomposition are essential for the decomposed element. \hfill \qed

There are obvious ‘$\kappa$-versions’ generalizing several of the results in this paper. Recall that $q$ is a $\kappa$-spoonhandle iff it generates a $\kappa$-directed principal filter. Calling a poset $\kappa$-spoonful if it has a cofinal subset of $\kappa$-spoonhandles, one easily verifies that $P - 1$ is $\kappa$-spoonful iff so is the cofinal subset $Q$.

**Theorem 7.** The top element of a $\lor$-pseudocomplemented poset $P$ has an irredundant join-decomposition into $\kappa$-prime elements iff $P - 1$ is $\kappa$-spoonful. In that case, for any maximal upper antichain $A$ of $\kappa$-spoonhandles, $A_*$ is the unique irredundant $\kappa$-prime decomposition of $1$ in $P$ and the unique direct decomposition into $\kappa$-prime atoms of $P_*$. The proof is easily accomplished with the help of Theorem 3 and Proposition 4. The case $\kappa = \infty$, where ‘$\kappa$-spoonful’ means ‘coatomic’, is one of Gorbunov’s theorems about canonical decompositions [20].

As expected, there are various applications of our general results to the situation of topological spaces – in fact, every lattice of closed sets is a coframe, hence a a complete (lower) Brouwerian lattice. By reasons of limited space, these and other applications to the decomposition theory for topological spaces and ideal systems of ordered sets are deferred to a separate note [15].

**References**


MARCEL ERNÉ, INSTITUT FÜR ALGEBRA, ZAHLENTHEORIE UND DISKRETEN MATHEMATIK, UNIVERSITÄT HANNOVER, GERMANY

E-mail address: erne@math.uni-hannover.de

URL: http://ww-iim.math.uni-hannover.de/html/person.html?id=82