Sober Spaces,
Well-Filtration and
Compactness Principles

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Abstract

We investigate various notions of sobriety for topological spaces that are equivalent in ZF set theory with the Axiom of Choice but may differ in the absence of choice principles. The Ultrafilter Principle UP (alias Prime Ideal Theorem) suffices for the desired conclusions. We derive from UP three topological postulates and prove their equivalence in ZF without choice: the Well-Filtration Principle, the Compactness Principle and the Irreducible Cutset Principle. On the other hand, the latter, applied to A(lexandroff)-spaces, immediately yields Rudin’s Lemma, while the other two, restricted to A-spaces, are equivalent to the statement that for any Noetherian poset, the ∧-semilattice of all finitely generated upper sets is again Noetherian.

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0 Introduction

Many mathematical statements that classically are derived from the Axiom of Choice (AC) already follow from or are even equivalent to the weaker Ultrafilter Principle (UP), requiring for any proper set-theoretical filter $F$ an ultrafilter containing $F$. It is well known that UP is equivalent to the Prime Ideal Theorem (PIT) for Boolean algebras or distributive lattices (see Scott [27] and Tarski [28] for early references). Perhaps less known is the fact that PIT is equivalent to the much more flexible

*Separation Lemma for Quantales: The complement of any Scott-open filter in a quantale is a lower set generated by prime elements*
In the present context, we shall need only the Separation Lemma for Locales (SL), alias Strong Prime Element Theorem [6], saying that for any Scott-open filter $U$ in a locale $L$ and any $a \in L \setminus U$, there is a $(\wedge\cdot)$prime $p \in L \setminus U$ with $a \leq p$. Recall that a locale or frame is a complete lattice $L$ enjoying the distributive law

$$a \wedge \bigvee B = \bigvee \{a \wedge b : b \in B\} \text{ for all } a \in L \text{ and } B \subseteq L,$$

that a subset $U$ is Scott-open iff

$$\bigvee D \in U \iff D \cap U \neq \emptyset \text{ for all directed subsets } D \text{ of } L,$$

that $U$ is a filter iff

$$\bigwedge F \in U \iff F \subseteq U \text{ for all finite subsets } F \text{ of } L,$$

and that an element $p$ is $(\wedge\cdot)$prime iff $\{a \in L : a \not\leq p\}$ is a filter.

The Separation Lemma for Quantales or Locales often provides very short proofs of algebraic, topological or order-theoretical statements whose derivation from AC would be more complicated.

As usual, we denote the Scott topology (consisting of all Scott-open sets) of a poset $P$ by $\sigma P$, and the resulting space by $\Sigma P$. The Lawson dual $\hat{P} = \delta P$ is the set of all Scott-open filters, ordered by inclusion. It plays a central role in the duality theory for continuous posets and semilattices (see [14, 15, 24]). For a recent choice-free proof of that duality, see [11].

Also in the present paper, we are working in ZF set theory without any choice principles, if not otherwise stated. We compare the classical concept of sobriety with another one that is equivalent to the former in ZF with UP but often more effective in concrete applications; we call it $\delta$-sobriety, referring to the Lawson dual $\delta P$. In ZF + UP, every sober space is well-filtered (see [15] and Section 1), but the point is that well-filtration follows from $\delta$-sobriety without invoking any choice principles.

We prove the equivalence of three consequences of the Sobriety Principle (requiring that every sober space is $\delta$-sober), namely: (FS) the Well-Filtration Principle, saying that every sober space is well-filtered, (CS) the Compactness Principle, stating that every filter base of compact saturated sets in a sober space has nonempty intersection, and (IS) the Irreducible Cutset Principle, saying that every filter base of compact saturated sets in an arbitrary space has an irreducible cutset. Then, we show that Rudin’s Lemma (a central tool in the theory of quasicontinuous posets; see [16] and [15]) is equivalent to the previous principles, restricted to locally supercompact spaces, or even to the smaller class of spaces with minimal bases. Restriction of the Well-Filtration Principle to the still much smaller class of Alexandroff spaces leads to the known but non-trivial fact that for any Noetherian poset, the semilattice of all finitely generated upper sets is again Noetherian. In the last section, we show that all choice principles considered here imply the Axiom of Choice for countable families of finite sets.
1 Variants of Sobriety

For any quasi-order \( \leq \) on a set \( X \) and for any subset \( Y \) of \( X \),
\[
\uparrow Y = \{ x \in X \mid y \leq x \text{ for some } y \in Y \}
\]
is the upper set or upset generated by \( Y \), and
\[
\downarrow Y = \{ x \in X \mid x \leq y \text{ for some } y \in Y \}
\]
is the lower set or downset generated by \( Y \). As usual, we write \( \downarrow y \) for the principal ideal \( \downarrow \{ y \} \) and \( \uparrow y \) for the principal filter \( \uparrow \{ y \} \).

For any topological space \( X \) with topology \( T = OX \) and for any point \( x \),
\[
T_x = \{ U \in T \mid x \in U \}
\]
is the open neighborhood filter at \( x \), and the specialization order is given by
\[
x \leq y \iff T_x \subseteq T_y \iff x \in \overline{\{ y \}}.
\]

\( \Sigma X \) denotes underlying set equipped with the specialization order; in the sequel, all order-theoretical statements about spaces refer to that quasi-order, if not otherwise stated. Thus, for \( Y \subseteq X \), the upper set generated by \( Y \) coincides with the saturation, the intersection of all open sets containing \( Y \) (whence the upper sets are the saturated ones), while the lower set generated by \( Y \) is the union of all closures of points in \( Y \); in particular, \( \overline{\{ y \}} = \downarrow \{ y \} \).

By definition, a space satisfies the \( T_0 \)-axiom iff the specialization order is antisymmetric, hence a partial order. A subset \( C \) is compact, respectively supercompact, if each directed, respectively arbitrary, open cover of \( C \) has a member containing \( C \) (no separation axiom required). Recall that a \( T_0 \)-space \( X \) is sober iff the only irreducible (i.e. \( \lor \)-prime) closed sets are the point closures. The \((\land \text{-})\)spectrum of a complete lattice \( L \) is the set \( P \) of all \((\land \text{-})\)prime elements \((\neq \lor L!)\), equipped with the hull-kernel topology
\[
T = \{ P \setminus \uparrow a : a \in L \}.
\]

It is well-known that the spectra of complete lattices are precisely the sober spaces (see, for example, [15, Ch.V-4]). In case \( P \) is meet-dense in \( L \), one speaks of a spatial locale. Via the open set functor in one direction and the spectrum functor in the other, the category of sober spaces is dual to the category of spatial locales (see [14] or [21]).

As usual, by a filter base we mean a system of sets that is directed by reverse inclusion and does not contain the empty set as a member. Two variants of the classical sobriety concept have turned out to be very useful in domain theory (cf. [14, 15]): call a \( T_0 \)-space \( X \)

- \( \delta \)-sober if each Scott-open filter \( \mathcal{V} \) of open sets in \( X \) (i.e. \( \mathcal{V} \in \delta OX \)) contains all open neighborhoods of \( \bigcap \mathcal{V} \),
- well-filtered if for any filter base \( \mathcal{B} \) of compact saturated sets, each (open) neighborhood of the meet \( \bigcap \mathcal{B} \) contains some member of \( \mathcal{B} \).
We shall need two more general definitions. Let \( \mathcal{Y} \) be a collection of saturated compact subsets (referred to as \( \mathcal{Y} \)-\textit{compact sets}) of a \( T_0 \)-space \( X \). We say \( X \) is \( \mathcal{Y} \)-well-filtered if for any filter base \( B \) of \( \mathcal{Y} \)-compact sets, \( \bigcap B \subseteq U \in \mathcal{O}X \) implies \( B \subseteq U \) for some \( B \in \mathcal{B} \), and we call the space locally \( \mathcal{Y} \)-\textit{compact} if each point has a neighborhood base consisting of members of \( \mathcal{Y} \) (not necessarily open ones). Our first lemma concerns the connections between \( \delta \)-sobriety and well-filtration in a choice-free setting (while the relationship between \( \delta \)-sobriety and classical sobriety will be discussed thereafter). For related work and an example of a well-filtered but non-sober space, see Kou [23].

**Lemma 1** (1) Every \( \delta \)-sober space is \( (\mathcal{Y}) \)-well-filtered.

(2) Conversely, every locally \( \mathcal{Y} \)-compact and \( \mathcal{Y} \)-well-filtered space is \( \delta \)-sober, provided \( \mathcal{Y} \) is closed under finite unions.

**Proof.** Let \( \mathcal{T} \) be the topology of the given space.

(1) If \( B \) is a filter base of compact saturated sets, then it is easy to see that the system
\[
\mathcal{V} = \{ U \in \mathcal{T} : \exists B \in \mathcal{B} (B \subseteq U) \}
\]
belongs to \( \delta \mathcal{T} \) and satisfies \( \bigcap B \subseteq \bigcap \mathcal{V} \). Hence, \( \bigcap B \subseteq U \in \mathcal{T} \) implies \( B \subseteq U \) for some \( B \in \mathcal{B} \).

(2) Consider any Scott-open proper filter \( \mathcal{V} \in \delta \mathcal{T} \) and put
\[
\mathcal{B} = \{ B \in \mathcal{Y} : \mathcal{V} \subseteq B \}
\]
We claim that \( \mathcal{B} \) is a filter base. For any finite subset \( \mathcal{E} \) of \( \mathcal{B} \), there is a \( V \in \mathcal{V} \) with \( V \subseteq \bigcap \mathcal{E} \). By local \( \mathcal{Y} \)-compactness, \( V \) is the union of the interiors of a directed system of \( \mathcal{Y} \)-compact subsets of \( V \). Being Scott open, \( \mathcal{V} \) must contain at least one of these interiors; thus, \( B \subseteq V \subseteq \bigcap \mathcal{E} \) for some \( B \in \mathcal{Y} \) with \( B^\circ \in \mathcal{V} \), and \( B \) is a lower bound of \( \mathcal{E} \) in \( \mathcal{B} \). By definition, \( \bigcap \mathcal{V} \subseteq \bigcap \mathcal{B} \). Assuming \( x \in \bigcap \mathcal{B} \setminus \bigcap \mathcal{V} \), pick a \( V \in \mathcal{V} \) with \( x \notin V \); then \( V \subseteq W = X \setminus \{ x \} \in \mathcal{T} \), whence \( W \notin \mathcal{V} \). Again by local \( \mathcal{Y} \)-compactness, \( \mathcal{D} = \{ C \in \mathcal{Y} : x \notin C \} = \{ C \in \mathcal{Y} : C \subseteq W \} \) is a directed system with \( \bigcup \{ C^\circ : C \in \mathcal{D} \} = W \), so there exists a \( C \in \mathcal{D} \) with \( C^\circ \in \mathcal{V} \), i.e. \( C \in \mathcal{B} \), which however leads to the contradiction \( x \in \bigcap \mathcal{B} \subseteq C \). Thus, \( \bigcap \mathcal{B} = \bigcap \mathcal{V} \).

Now, if \( X \) is \( \mathcal{Y} \)-well-filtered, \( \bigcap \mathcal{V} = \bigcap \mathcal{B} \subseteq U \in \mathcal{T} \) implies \( B \subseteq U \) for some \( B \in \mathcal{B} \), hence \( B^\circ \subseteq \mathcal{V} \), \( B^\circ \subseteq U \) and therefore \( U \in \mathcal{V} \), proving \( \delta \)-sobriety. \( \square \)

In [17], Hofmann and Mislove studied the poset \( \mathcal{Q}X \) of all (quasi-)compact saturated sets (ordered by \textit{dual} inclusion) of arbitrary \( T_0 \)-spaces. The theorem below includes some of their results and the precise relationships between sobriety, \( \delta \)-sobriety and well-filtration, established first in [5] and submitted to authors of the \textit{Compendium of Continuous Lattices} [14] in 1984. With exception of the role of the involved choice principles, these results were collected together in Chapter II-1 of the 2003 edition \textit{Continuous Lattices and Domains} [15]. For the reader’s convenience, we add here a short proof, pointing to the only place where a choice principle is involved.
Theorem 1 The following conditions on a $T_0$-space $X$ are equivalent:

(a) $X$ is $\delta$-sober.

(b) $\delta \mathcal{O}X$ consists of all open neighborhood filters of compact sets.

(c) The map $\mathcal{V} \mapsto \bigcap \mathcal{V}$ is an isomorphism between $\delta \mathcal{O}X$ and $\mathcal{Q}X$.

Each of these conditions implies sobriety, and the converse implication is assured by the Ultrafilter Principle. Furthermore, (a)–(c) imply that

(d) $X$ is well-filtered.

In locally compact spaces, all four conditions (a)–(d) are equivalent, and in spaces with a base of compact open sets, they are also equivalent to

(e) Each filter $\mathcal{B}$ of compact open sets contains any open $U$ with $\bigcap \mathcal{B} \subseteq U$.

Proof. (a)⇒(c). Put $T = \mathcal{O}X$. By hypothesis (a), the map

$$\Phi : \mathcal{Q}X \to \delta T, \ C \mapsto \{U \in T : C \subseteq U\}$$

is left inverse to the map

$$\Psi : \delta T \to \mathcal{Q}X, \ \mathcal{V} \mapsto \bigcap \mathcal{V},$$

which is well-defined, because $\bigcap \mathcal{V}$ is saturated and compact (indeed, if $D$ is a directed subset of $T$ with $\bigcap \mathcal{V} \subseteq \bigcup D$ then $\bigcup D \in \mathcal{V}$, and as $\mathcal{V}$ is Scott open, there is a $U \in \mathcal{V} \cap D$, a fortiori $\bigcap \mathcal{V} \subseteq U$). For any $C \in \mathcal{Q}X$, the equation $C = \bigcap \Phi(C) = \Psi \circ \Phi(C)$ is clear since $C$ is saturated.

For (c)⇒(b), note that $C$ is compact iff so is $\uparrow C = \bigcap \Phi(C)$, and that $\Phi(C) = \Phi(\uparrow C)$. Clearly, (b) implies (a): $\bigcap \mathcal{V} \subseteq U \in T$ ⇒ $U \in \Phi \circ \Psi(\mathcal{V}) = \mathcal{V}$.

(b) implies sobriety: For any prime open set $P$, the system $\mathcal{V}$ of all $U \in T$ with $U \not\subseteq P$ is a Scott-open filter, since $T \setminus \mathcal{V}$ is a principal ideal of $T$. Hence $\mathcal{V} = \Phi(C)$ for some $C \in \mathcal{Q}X$, and $P \not\in \Phi(C)$, i.e. $C \not\subseteq P$. Choosing $x \in C \setminus P$, one obtains $P \subseteq U = X \setminus \{x\} \in T$. On the other hand, $x \not\in U$ yields $U \not\in \mathcal{V}$ (otherwise $x \in C \subseteq U$), and so $U \subseteq P$. Thus, $U = P$ is the complement of a point closure.

$\text{SL}$, or equivalently $\text{UP}$, together with sobriety of $X$ implies (a):

Consider any $\mathcal{V} \in \delta T$ and put $C = \bigcap \mathcal{V}$. If $\mathcal{M} = \{U \in T \setminus \mathcal{V} : C \subseteq U\}$ would be nonempty, the Separation Lemma would give a prime open set $P \in \mathcal{M}$. But sobriety requires that $P$ is the complement of a point closure $\{x\}$; if $x \not\in \mathcal{V}$ for some $\mathcal{V} \in \delta T$ then $V \subseteq P \in \mathcal{V}$, which is impossible, while $x \not\in \bigcap \mathcal{V}$ leads to $x \in C \subseteq P$, again a contradiction. Hence $\mathcal{M}$ must be empty, i.e. $C \subseteq U \in T$ entails $U \in \mathcal{V}$.

The implication (a)⇒(d) and the reverse implication for locally compact spaces are special instances of Lemma 1. A similar argument shows that (e) implies (a) if $X$ has a base of compact open sets (details are in [5]), and clearly (d) implies (e).
We have seen that under suitable choice principles weaker than AC, both sobriety concepts are equivalent, but the precise strength of that coincidence remains vague. Our considerations at the end of this paper will demonstrate that at least the Axiom of Choice for countable families of finite sets is necessary. At the other extreme, it might be (but not very likely) that the above coincidence is tantamount to the full Separation Lemma for arbitrary locales. What we can prove is an intermediate result:

**Proposition 1** Let $X$ be a class of sober spaces and $X'$ the corresponding class of locales isomorphic to topologies of spaces in $X$. The following two principles are equivalent in ZF (and consequences of UP):

**SX** The Sobriety Principle for Spaces in $X$:
Every space in $X$ (every spectrum of a locale in $X'$) is $\delta$-sober.

**SX'** The Separation Lemma for Locales in $X'$:
In every locale that belongs to $X'$, any lower set complementary to a Scott-open filter is generated by primes.

**Proof.** In view of the earlier remarks about spectra, it remains to verify that $\delta$-sobriety of the spectrum $(P, T)$ of a spatial locale $L$ is equivalent to the Separation Lemma for that locale. Recall that the hull-kernel topology $T$ is isomorphic to $L$ via the assignment $a \mapsto P \uparrow a$. Hence, for any $U \in T$ and any Scott-open filter $V \in \delta T$, there are $u \in L$ and $V \in \delta L$ such that $U = P \uparrow u$ and $V = \{P \uparrow v : v \in V\}$. Moreover, $U \in V$ is equivalent to $u \in V$, and $\bigcap V \subseteq U$ means $P \cap \uparrow u \subseteq \uparrow V = V$. Thus, the contraposition of the defining implication for $\delta$-sobriety,

$V \in \delta T$ and $U \in T \setminus V \Rightarrow \bigcap V \not\subseteq U$

is equivalent to the requirement that for each $V \in \delta L$ and $u \in L \setminus V$, there is a prime $p \in P$ with $u \leq p$ but $p \not\in V$. But that is exactly the statement of the Separation Lemma. \qed

In passing, we note the following complementary formulation of SX':

*Each Scott-open filter in a locale that belongs to $X'$ contains with any closed subset of the spectrum its meet.*

Combining Proposition 1 with Theorem 1, we see that the Separation Lemma for Spatial Locales (SS') suffices to prove well-filtration of all sober spaces. It could be the case that in ZF without choice, the Sobriety Principle for arbitrary spaces (SS), though being equivalent to SS', is strictly weaker than SL, the Separation Lemma for arbitrary locales, but strictly stronger than the postulate that every sober space is well-filtered.
2 Monotone convergence spaces

A monotone convergence space is a $T_0$-space in which every monotone net has a supremum and converges to it (see e.g. [15, II-3]). In particular, every monotone convergence space is up-complete (a dcpo) relative to the specialization order. The next proposition shows that not only every sober space, but also every well-filtered space is a monotone convergence space, and that both implications hold in ZF without choice.

**Proposition 2** For a $T_0$-space $X$, the following are equivalent:

(a) $X$ is a d-space: The closure of any directed subset is a point closure.
(b) $X$ is temperate: $P = \Sigma X$ is a dcpo, and $OX$ is coarser than the Scott topology $\sigma P$.
(c) $X$ is a monotone convergence space: Every monotone net in $X$ has a supremum to which it converges.
(d) $X$ is $C$-well-filtered for the set $C$ of all cores.

**Proof.** The equivalence of (a), (b) and (c) is folklore (see [10, 14, 15, 29]), and its proof is an easy exercise.

(b)$\Rightarrow$(d). Any filter base of cores is of the form $B = \{\uparrow d : d \in D\}$ for a (unique) directed subset $D$ of $P = \Sigma X$. If $\bigcap B \subseteq U$ for some $U \in OX$ then $\bigvee D \in U \in \sigma P$, hence $D \cap U \neq \emptyset$, which means $B \subseteq U$ for some $B \in B$.

(d)$\Rightarrow$(b) is obtained similarly. In order to see that each directed subset $D$ of $P$ has a join, consider the filter base $B = \{\uparrow d : d \in D\}$ and the closed set $A = \bigcap \{\downarrow y : D \subseteq \downarrow y\}$ (which is an intersection of point closures). Each $B \in B$ intersects $A$ (as $D$ is contained in $A$), and so $\bigcap B$ meets $A$. But the only element of $\bigcap B \cap A$ is the join (supremum) of $D$.

The terminology “d-space” or “temperate space” is due to Wyler [29]. Not all up-complete posets arise from well-filtered spaces; in order to verify that claim, let us revisit a famous example due to Johnstone [20]:

**Example 1** The set $\omega \times (\omega \cup \{\omega\})$, ordered by

$$(m, n) \subseteq (m', n') \iff (m = m' \text{ and } n \leq n') \text{ or } (n \leq m' \text{ and } n' = \omega)$$

is up-complete, but there is no sober topology with specialization order $\subseteq$, because the whole set has to be irreducible (see [20] or [8, Ex.6.13]). Every nonempty Scott-open set must contain a point $(m, n)$ with $n < \omega$, hence all maximal elements $(m', \omega)$ with $m' \geq n$. Therefore,

$$B = \{(m, \omega) : m \geq n \} : n \in \omega$$

is a filter base of nonempty compact saturated sets with empty intersection. Thus, no space with specialization order $\subseteq$ and a topology coarser than the Scott topology can be well-filtered.
Associating with any poset $P$ its Scott space $\Sigma P$, the underlying set equipped with the Scott topology, one obtains a functor $\Sigma$ from the category of posets and maps preserving directed joins to the category of topological spaces, and $\Sigma$ is right inverse to the specialization functor $\Sigma^{-}$ in the opposite direction. The restriction of $\Sigma^{-}$ to sober spaces yields a functor to the category of dcpos, but that functor is not onto on objects, as witnessed by Example 1. Wyler [29] writes “wir machen aus der Not eine Tugend” and proposes to study, instead of sober spaces, the larger class of all d-spaces. He justifies his approach by showing that the restriction of $\Sigma$ to up-complete posets is left adjoint to the restriction of $\Sigma^{-}$ to d-spaces, and that some important properties of sober spaces extend to d-spaces.

While there are even complete lattices that fail to be sober in their Scott topology (see Isbell [18]), the upper topology (the weakest topology making all principal ideals closed) behaves better in that respect:

**Lemma 2** The Axiom of Choice for families of finite sets ensures that the up-complete $\lor$-semilattices are those which are sober in their upper topology.

**Proof.** Every closed set relative to the upper topology is of the form $A = \bigcap \{ \downarrow F_i : i \in I \}$ for a family of finite subsets $F_i$. If $A$ is $\lor$-prime then the inclusions $A \subseteq \downarrow F_i = \bigcup \{ \downarrow x : x \in F_i \}$ lead to a choice function $\chi \in \prod_{i \in I} F_i$ with $A \subseteq \downarrow \chi(i)$ for all $i \in I$, whence

$$A = \bigcap \{ \downarrow F_i : i \in I \} = \bigcap \{ \downarrow \chi(i) : i \in I \} = \downarrow \bigwedge \{ \chi(i) : i \in I \}$$

is a point closure. \hfill $\Box$

Johnstone’s counterexample demonstrates that Lemma 2 fails for up-complete posets. Since the Ultrafilter Principle entails the Axiom of Choice for finite sets, we arrive at

**Theorem 2** In ZF with UP, the following attributes for a $\lor$-semilattice, equipped with the upper topology, are all equivalent:

- up-complete, temperate, d-space, well-filtered, sober, $\delta$-sober.

It remains open whether the conclusion in Lemma 2 actually requires choice. In certain concrete instances, sobriety may be established directly.

**Example 2** For power set lattices $L = \mathcal{P}X$, sobriety in the upper topology means that the power sets are the only $\cup$-irreducible systems of finite character. Indeed, the Scott-closed subsets of power sets are just the systems of finite character (containing a set if and only if all of its finite subsets belong to the system). Assume now that $\mathcal{X}$ is a system of finite character distinct from the power set $\mathcal{P}X$, where $X = \bigcup \mathcal{X}$. Then there is a finite set $F \in \mathcal{P}X \setminus \mathcal{X}$, and a short computation yields $\mathcal{X} \subseteq \bigcup \{ \mathcal{P}(X \setminus \{x\}) : x \in F \}$ but $\mathcal{X} \not\subseteq \mathcal{P}(X \setminus \{x\})$ for all $x \in F$. Hence, $\mathcal{X}$ cannot be $\lor$-prime in the lattice of Scott-closed subsets.
3 Compactness Principles and Rudin’s Lemma

As we shall see below, the well-filtration of sober spaces is tantamount to a seemingly more general principle concerning irreducible sets in arbitrary topological spaces. Besides its applications to topology and order theory, this principle may also have interesting consequences for the foundations of algebraic geometry, where the irreducible sets play a central role in the interaction between algebraic, geometric and topological concepts.

Recall that $X^s$, the sob(e)riﬁcation of a $T_0$-space $X$, is the spectrum of the lattice of open sets (see [15, Ch.V] or [21, Ch.II]). Up to homeomorphism, the sobriﬁcation is the unique sober space containing $X$ as a strictly dense subspace, i.e. each closed set in $X^s$ has a dense subset contained in $X$ (cf. [15, Ch.V-5]). In particular, associating with each closed set in $X$ its closure in $X^s$ yields an isomorphism between the closed set lattices of $X$ and of $X^s$.

**Proposition 3** Let $X$ be a class of topological spaces. The following three statements are consequences of the Separation Lemma for (Spatial) Locales, hence of the Ultrafilter Principle:

- **FX** The Well-Filtration Principle:
  Every sober space in $X$ is well-filtered.

- **CX** The Compactness Principle:
  Every filter base of compact saturated subsets of a sober space in $X$ has nonempty intersection.

- **IX** The Irreducible Cutset Principle:
  For any filterbase $\mathcal{B}$ of compact saturated subsets of a space in $X$, there is an irreducible (closed) subset meeting each member of $\mathcal{B}$.

The implications $\text{FX} \Rightarrow \text{CX}$ and $\text{IX} \Rightarrow \text{CX}$ are generally valid.

If $X$ is stable under the formation of closed subspaces then $\text{FX}$ and $\text{CX}$ are equivalent, and if $X$ is stable under sobriﬁcation then $\text{CX}$ and $\text{IX}$ are equivalent. Hence, all three statements are equivalent for the class $X = S$ of arbitrary topological spaces.

**Proof.** $\text{FX} \Rightarrow \text{CX}$ is an obvious specialization.

$\text{IX} \Rightarrow \text{CX}$. The only irreducible closed sets in a sober space are the point closures, and if a (compact) saturated set meets a point closure $\{x\}$ then it must already contain the point $x$.

$\text{CX} \Rightarrow \text{FX}$. Assume $\mathcal{B}$ is a filter base of compact saturated subsets of a sober space $X$ in $X$, and $U$ is an open set with $B \nsubseteq U$ for all $B \in \mathcal{B}$. Then $\{B \setminus U : B \in \mathcal{B}\}$ is a filter base of compact saturated sets in the closed subspace $A = X \setminus U$, which is sober, too (indeed, any irreducible closed set in $A$ is also irreducible and closed in $X$, hence the closure of a point that
must belong to $A$). Under the hypothesis that $A$ is a member of $X$, we get (by $CX$) a common point of all $B \setminus U$ with $B \in \mathcal{B}$, whence $\bigcap \mathcal{B}$ cannot be contained in $U$. By contraposition, $\bigcap \mathcal{B} \subseteq U$ entails $B \subseteq U$ for some $B \in \mathcal{B}$.

Let $X$ be a space in $X$ (which may be assumed to be $T_0$, by passing to the $T_0$-reflection if necessary), and let $\mathcal{B}$ be a filter base of compact saturated subsets. Then, their saturations in $X^s$ build a filter base, too. Hence, if $X^s$ is also a member of $X$ then, by $CX$, these saturations (being compact in $X^s$) have a common point $y \in X^s$; the trace $C = X \cap \{y\}$ of the closure $\overline{\{y\}}$ formed in $X^s$ is then a closed set in $X$ meeting each $B \in \mathcal{B}$ (indeed, $y \in \uparrow B$ implies $B \cap C = B \cap \overline{\{y\}} \neq \emptyset$), and $C$ is irreducible in $X$ since for closed $A$ in $X$, $C \subseteq A \iff \overline{C} \subseteq \overline{A} \iff y \in \overline{A}$ (closures in $X^s$).

Specifically, the previous proposition applies to the following fundamental classes of topological spaces (see [7] and [10]), which are stable both under sobrification (because they may be defined by a lattice-theoretical property of their topologies) and under the formation of closed subspaces:

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<th>Category</th>
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<tr>
<td>$sA$</td>
<td>sober $A$-spaces</td>
<td>$A'$</td>
<td>superspatial</td>
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<td></td>
<td>sober spaces in which all cores are open</td>
<td></td>
<td>the $\wedge$-primes are meet dense and $\wedge$-prime</td>
</tr>
<tr>
<td>$(s)B$</td>
<td>(sober) $B$-spaces</td>
<td>$B'$</td>
<td>superalgebraic</td>
</tr>
<tr>
<td></td>
<td>have a base of open cores (+sober)</td>
<td></td>
<td>the $\wedge$-prime elements are meet dense</td>
</tr>
<tr>
<td>$(s)C$</td>
<td>(sober) $C$-spaces</td>
<td>$C'$</td>
<td>supercontinuous</td>
</tr>
<tr>
<td></td>
<td>have neighborhood bases of cores (+sober)</td>
<td></td>
<td>completely distributive: $a = \bigvee \bigcap {\downarrow B : a \leq \bigvee B}$</td>
</tr>
</tbody>
</table>

The core of a point is the intersection of its neighborhoods, i.e. the principal filter generated by that point. Equivalently, the cores are precisely the supercompact saturated sets, whence $C$-spaces are also called locally supercompact. While spatiality of superalgebraic locales is trivial, the spatiality of supercontinuous lattices probably requires something like the Principle of Dependent Choices ($DC$). As explained in [7], [10] and [11], the above three categories of spaces admit purely order-theoretical descriptions:

Lemma 3 Equipped with the Scott topology,
- the Noetherian posets are precisely the $(\delta)$-sober $A$-spaces,
- the algebraic domains are precisely the $(\delta)$-sober $B$-spaces,
- and the $\delta$-domains are precisely the $(\delta)$-sober $C$-spaces.

Here, a $\delta$-domain is a $dcpo$ $P$ such that for each $y \in P$, the set of all $x$ with $y \in V \subseteq \downarrow x$ for some $V \in \delta P$ is directed with join $y$; $DC$ ensures that the $\delta$-domains are just the continuous ones [15].
The category $\mathbf{A}$ of all $\mathbf{A}$-spaces is not stable under sobrification (in fact, the sobrifications of $\mathbf{A}$-spaces are the sober $\mathbf{B}$-spaces, that is, the algebraic domains)! While every subspace of an $\mathbf{A}$-space is again an $\mathbf{A}$-space, every topological space occurs as a subspace of a $\mathbf{B}$-space, as shown in [10].

The Irreducible Cutset Principle says that the grill

$$B^\# = \{ A \subseteq \bigcup B : \forall B \in B (A \cap B \neq \emptyset) \}$$

of any filter base $B$ of compact saturated sets has an irreducible member. Let us apply this to the special class of Alexandroff discrete spaces ($\mathbf{A}$-spaces). From the original work of Alexandroff [1] we know that these are just the quasi-ordered sets, topologized by the system of all upper sets. In that specific situation, the compact saturated sets are precisely the finitely generated upper sets, and the irreducible subsets are the directed ones, whence the closed irreducible sets are exactly the ideals (i.e., the directed lower sets).

Thus, the Irreducible Cutset Principle for $\mathbf{A}$-spaces ($\mathbf{IA}$) is equivalent to

\textbf{RL} Rudin’s Lemma (on Filter Bases of Finitely Generated Upper Sets):

For any collection $\mathcal{E}$ of finite sets in a poset such that $\{ \uparrow E : E \in \mathcal{E} \}$ is a filter base, there is a directed subset of $\bigcup \mathcal{E}$ meeting each $E \in \mathcal{E}$.

In other words, the grill $\mathcal{E}^\#$ contains a directed set. Indeed, that directed set may be chosen as an ideal (irreducible closed subset) of the induced poset with underlying set $U = \bigcup \mathcal{E}$ (because $\{ \uparrow E \cap U : E \in \mathcal{E} \}$ is a filter base on $U$).

Originally, Rudin [26] proved her lemma by transfinite methods, using $\mathbf{AC}$. This lemma plays a crucial role in the theory of \textit{quasicontinuous posets or domains}, developed by Gierz, Lawson and Stralka (see [16] and [15]). The proof of $\mathbf{RL}$ was simplified in [15, Ch.IV] but still involved the full strength of $\mathbf{AC}$. Our approach instead is based on the weaker $\mathbf{UP}$, or, more directly, on the (possibly still weaker) Separation Lemma for Spatial Locales. Moreover, the general version in form of the Irreducible Cutset Principle for arbitrary spaces opens applications to a considerably wider range of problems. We shall use the following self-refinement of Rudin’s Lemma:

\textbf{RL′} Let $\mathcal{E}$ be a set of finite subsets of a poset such that $\{ \uparrow E : E \in \mathcal{E} \}$ is a filter base. Then each lower set $A$ that meets all members of $\mathcal{E}$ contains a directed set $D \in \mathcal{E}^\#$. Hence, if $U$ is an upper set not containing any member of $\mathcal{E}$ then there is a directed $D \in \mathcal{E}^\#$ with $U \cap D = \emptyset$.

For the implication $\mathbf{RL} \Rightarrow \mathbf{RL′}$, apply $\mathbf{RL}$ to the filter base consisting of all $\uparrow (E \cap A)$ with $E \in \mathcal{E}$. For the second part of $\mathbf{RL′}$, pass to complements. Of course, $\mathbf{RL}$ is a special instance of $\mathbf{RL′}$ ($U = \emptyset$).

We say a subset of a space is \textit{hypercompact} if its saturation is a finitely generated upper set. Obviously, every hypercompact set is compact, but the converse fails in general. Let $\mathcal{H}$ denote the collection of all hypercompact
saturated sets (that is, of all finitely generated upper sets). Summarizing
the previous thoughts, we now are ready for

**Theorem 3** The following principles are equivalent in ZF set theory:

RL Rudin’s Lemma.

IA The Irreducible Cutset Principle for A-spaces.

SB’ The Separation Lemma for Superalgebraic Locales.

SB The Sobriety Principle for B-spaces.

FB The Well-Filtration Principle for B-spaces.

CB The Compactness Principle for B-spaces.

IB The Irreducible Cutset Principle for B-spaces.

SC’ The Separation Lemma for Supercontinuous Spatial Locales.

SC The Sobriety Principle for C-spaces.

FC The Well-Filtration Principle for C-spaces.

CC The Compactness Principle for C-spaces.

IC The Irreducible Cutset Principle for C-spaces.

TH Every temperate (monotone convergence) space is \( H \)-well-filtered.

All of these statements follow from the Ultrafilter Principle.

**Proof.** RL \( \Rightarrow \) TH. If \( B \) is a filter base of finitely generated upper sets and
\( U \) is an open (hence upper) set with \( B \not\subseteq U \) for all \( B \in B \) then, by RL’,
there is a directed \( D \in B^\# \) with \( D \cap U = \emptyset \). On the other hand, we know
from Proposition 2 that \( x = \bigvee D \) exists and lies in \( \bigcap B \), whereas \( x \) cannot
belong to the Scott-open set \( U \). Thus, \( \bigcap B \not\subseteq U \), as desired.

TH \( \Rightarrow \) SC. C-spaces are locally supercompact, *a fortiori* locally hyper-
compact (i.e. locally \( H \)-compact). Now, Lemma 1 applies to show that any
\( H \)-well-filtered C-space is \( \delta \)-sober. But sober spaces are temperate spaces,
and these are \( H \)-well-filtered by hypothesis TH.

The remaining implications and equivalences have been established earlier
(see Propositions 1 and 3):

\[
\begin{align*}
\text{TH} & \Rightarrow \text{SC'} \iff \text{SC} \Rightarrow \text{FC} \iff \text{CC} \iff \text{IC} \\
& \Downarrow \\
\text{SB'} & \iff \text{SB} \Rightarrow \text{FB} \iff \text{CB} \iff \text{IB} \Rightarrow \text{IA} \Rightarrow \text{RL} \Rightarrow \text{TH}.
\end{align*}
\]

An obvious question is whether in the above statements, \( B \) may be replaced
with A in order to obtain a few more equivalent conditions. Of course,
in the “classical” situation ZF+AC, all of these statements are not only
equivalent but *provable*. However, in ZF without choice, we can only say
that the “A-versions” are mutually equivalent: see the next section.
4 Sober A-spaces and Noetherian Posets

This section is devoted to an investigation of equivalents of the Compactness Principle for A-spaces and its relations to maximality and chain conditions in posets $P$. In ZF without choice, one has the trivial implications

\[ P \text{ is co-well-founded}: \text{every nonempty subset has a maximal element} \]

\[ \Downarrow \]

\[ P \text{ is Noetherian}: \text{every directed subset has a greatest element} \]

\[ \Downarrow \]

\[ \text{equivalently, every ideal is principal} \]

\[ P \text{ satisfies the acc}: \text{every ascending chain has a greatest member} \]

and it is well-known that the Principle of Dependent Choices (DC) makes all three properties equivalent. A much less trivial fact is that (under suitable choice principles) these properties are transferred from a poset $P$ to $S = \mathcal{F}_\Lambda P$, the semilattice of all finitely generated upper sets, ordered by reverse inclusion. The dual result was established by a rather complicated proof in the 1948 edition of Birkhoff’s Lattice Theory [3], and later by simpler arguments in [4] and [13]. The proof in [4] is based on the following useful Induction Principle, valid in all co-well-founded posets.

**Induction Principle for Finitely Generated Upper Sets:**

If a subset $\mathcal{Y}$ of $S = \mathcal{F}_\Lambda P$ satisfies

(i) $\{Y \in S : Y \subseteq \uparrow x\} \subseteq \mathcal{Y} \Rightarrow \uparrow x \in \mathcal{Y}$

(ii) $Y \in S$ and $\{\uparrow x : x \in Y\} \subseteq \mathcal{Y} \Rightarrow Y \in \mathcal{Y}$

then $\mathcal{Y}$ contains all finitely generated upper sets.

**Proposition 4** Consider the following conditions on a poset $P$:

(a) $P$ enjoys the Induction Principle for Finitely Generated Upper sets.

(b) $P$ is co-well-founded. \hspace{1cm} (b’) $\mathcal{F}_\Lambda P$ is co-well-founded.

(c) $P$ is Noetherian. \hspace{1cm} (c’) $\mathcal{F}_\Lambda P$ is Noetherian.

(d) $P$ satisfies the acc. \hspace{1cm} (d’) $\mathcal{F}_\Lambda P$ satisfies the acc.

In ZF, all implications sketched in the diagram below hold true:

\[
\begin{array}{ccc}
(a) & \Rightarrow & (b) \\
& \Rightarrow & \ \\
& \Rightarrow & (c) \\
& \Rightarrow & \ \\
& \Rightarrow & (d)
\end{array}
\]

In ZF + DC, all seven conditions are equivalent.
Outline of Proof (in order to see that no choice principles are involved).

(a)⇒(b). Apply the Induction Principle to the set $\mathcal{Y}$ of all co-well-founded members of $\mathcal{S}$.

(b)⇒(a). Assume $\mathcal{Y} \neq \mathcal{S}$; then, by (b) and (ii), there is a maximal $x_0$ such that $\uparrow x_0 \notin \mathcal{Y}$. But then, applying (i) and (ii) once again, one would obtain an $x > x_0$ with $\uparrow x \notin \mathcal{Y}$.

(a)⇒(d'). Apply the Induction Principle to the set $\mathcal{Y}$ of all elements of $\mathcal{S}$ that are not the first member of a properly ascending (i.e. $\subseteq$-descending) sequence in $\mathcal{S}$. Then (i) is obvious. For (ii), consider any $Y_0 \in \mathcal{S}\setminus \mathcal{Y}$ and the finite set $F_0$ of its minimal elements. By the choice of $Y_0$, there is a sequence $Y_0=\uparrow F_0 \supset Y_1 = \uparrow F_1 \supset Y_2 = \uparrow F_2 \ldots$ in $\mathcal{S}$. For each $x \in F_0$, define recursively a sequence $Y_0(x) \supset Y_1(x) \supset Y_2(x) \ldots$ in $\mathcal{S}$ by

$$Y_0(x) = \uparrow x, \quad Y_{n+1}(x) = \uparrow (F_{n+1} \cap Y_n(x)).$$

A straightforward induction yields (⋆) $Y_n = \bigcup \{Y_n(x) : x \in F_0\}$ for all $n$. 

Now, if $\uparrow x \in \mathcal{Y}$ for all $x \in Y_0$ then each of the sets $\{Y_n(x) : n < \omega\}$ and so, by (⋆), the set $\{Y_n : n < \omega\}$ would be finite, i.e. $Y_0 \in \mathcal{Y}$, a contradiction. The other implications in the diagram are clear, and DC entails the remaining implications (d)⇒(b) and (d')⇒(b').

\[\square\]

**Theorem 4** The following principles are equivalent in ZF set theory:

- **NL** The Noetherian Lift Lemma: If $P$ is Noetherian then so is $F_0 P$.
- **SA′** The Separation Lemma for Superspatial Locales.
- **SA** The Sobriety Principle for $A$-spaces.
- **CA** The Compactness Principle for $A$-spaces.
- **IsA** The Irreducible Cutset Principle for sober $A$-spaces.

These statements are consequences of Rudin’s Lemma, hence of the Ultrafilter Principle; but they also follow from the Principle of Dependent Choices.

**Proof.** The equivalence of **SA** and **SA′** follows from Proposition 1, and that of **CA**, **FA** and **IsA** from Proposition 3.

**FA**, translated into order-theoretical terms (passing from $T_0$-$A$-spaces to posets $P$), says that for every directed subset $\mathcal{B}$ of $F_0 P$, any upper set $U$ with $\bigcap \mathcal{B} \subseteq U$ contains some member of $\mathcal{B}$. But the latter simply means (as $\bigcap \mathcal{B}$ itself is an upper set) that $\mathcal{B}$ has a greatest (!) element in the order of $F_0 P$. Thus, **FA** is tantamount to **NL**. Finally, **FA** is equivalent to **SA**, since $A$-spaces are locally (super)compact, so that by Lemma 1, well-filtration is equivalent to $\delta$-sobriety.  

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Note that \(UP\) and \(DC\) are independent postulates in \(ZF\) (see e.g. Jech [19]), and that both together are still weaker than \(AC\) (see Pincus [25]).

Summing up our previous deductions, we obtain the following implication diagram, which is perhaps not yet complete (that is, some further implications might hold in \(ZF\)):

\[
\begin{array}{c}
\text{AC} \\
\downarrow \\
\uparrow \downarrow \\
\text{UP} \leftrightarrow \text{SL} \\
\downarrow \\
\text{SS}' \leftrightarrow \text{SS} \leftrightarrow \text{FS} \leftrightarrow \text{CS} \leftrightarrow \text{IS} \leftrightarrow \text{RL}' \\
\downarrow \\
\text{SC}' \leftrightarrow \text{SC} \leftrightarrow \text{FC} \leftrightarrow \text{CC} \leftrightarrow \text{IC} \leftrightarrow \text{TH} \\
\downarrow \\
\text{SB}' \leftrightarrow \text{SB} \leftrightarrow \text{FB} \leftrightarrow \text{CB} \leftrightarrow \text{IB} \leftrightarrow \text{IA} \\
\downarrow \\
\text{SA}' \leftrightarrow \text{SA} \leftrightarrow \text{FA} \leftrightarrow \text{CA} \leftrightarrow \text{IsA} \leftrightarrow \text{NL} \\
\end{array}
\]

We have mentioned that, via the specialization functor, the sober \(A\)-spaces are in one-to-one correspondence with the Noetherian posets, and via the open set functor, they correspond to the superspatial locales, in which the \(\land\)-primes are not only meet-dense but also (completely) \(\land\)-prime. We supplement these facts by establishing the analogous correspondences for co-well-founded posets and for poset with acc, respectively. To that aim, let us recall that a space is scattered if each nonempty subset has an isolated point, and a lattice is strongly atomic if each non-trivial interval has an atom.

By a \(c_\omega\)-space we mean a \(T_0\)-space in which the closure of any \(\omega\)-chain (i.e. any ascending sequence relative to specialization) is a point closure, and we say a locale is \(c_\omega\)-spatial iff it is isomorphic to the topology of a \(c_\omega\)-space; for \(A\)-topologies, the latter means that every descending chain of \(\land\)-primes has a least element. The following facts are now easily accomplished (cf.[7]):

**Lemma 4** For a \(T_0\)-\(A\)-space \(X\) and the corresponding poset \(P = \Sigma^-X\), one has the following equivalences:

- \(P\) is co-well-founded \(\iff\) \(X\) is a scattered space \(\iff\) \(\mathcal{O}X\) is strongly atomic
- \(P\) is Noetherian \(\iff\) \(X\) is a (sober) \(d\)-space \(\iff\) \(\mathcal{O}X\) is superspatial
- \(P\) satisfies the acc \(\iff\) \(X\) is a \(c_\omega\)-space \(\iff\) \(\mathcal{O}X\) is \(c_\omega\)-spatial

In \(ZF+DC\), all nine statements are equivalent.
5 König’s Infinity Lemma

The equivalence theorems derived in the previous sections would not be really substantial if the involved statements (not only their equivalence) could be established in ZF set theory without choice. But that is not the case, as shall be demonstrated in this final section.

Call a (binary) relation $R$ locally finite if the sets

$$xR = \{y : xRy\}$$

are finite for all $x$ (in the domain). The powers of $R$ are defined, as usual, as iterated relation products. The covering relation $R$ of a poset is given by $xRz$ iff $x < z$ but no $y$ satisfies $x < y < z$; if its transitive-reflexive closure is the whole order relation, we speak of a concatenated poset. By a tree, we mean a down-directed poset in which all (principal) ideals are chains, and by an $\omega$-tree, a poset with least element such that all ideals are even $\omega$-chains and the covering relation is locally finite, i.e. each element has only a finite numbers of covers. Dénes König [22] was one of the first mathematicians to investigate such trees; today, they play a crucial role in various fields of logic, set theory, graph theory and computer science.

Example 3 $\omega$-trees of words (cf. Jech [19, p.115]).

Let $S$ be a set and $S^*$ the set of all finite sequences (“words”) in $S$. If $T$ is a lower set in $S^*$ such that for each $t \in T$, the set $\{s \in S : (t,s) \in T\}$ is finite, then $T$ is an $\omega$-tree (with the empty word as root).

This becomes evident from an alternate characterization of $\omega$-trees:

Lemma 5 The $\omega$-trees are precisely the concatenated locally finite trees.

Proof. That an $\omega$-tree is concatenated follows from the fact that each principal ideal must be a finite chain. Conversely, assume $T$ is a concatenated tree with locally finite covering relation. Let $D$ be an ideal of $T$. Any two elements of $D$ lie in a common principal ideal and are therefore comparable; thus, $D$ is a chain. If $D$ has a greatest element, it is a finite chain (by the concatenation property), and we are done. Otherwise, define recursively elements $d_n \in D$ by

$$d_n = \min\{d \in D : d_k < d \text{ for } k < n\}.$$

It remains to verify that $D = \{d_n : n < \omega\}$. By the recursive construction, $d_n$ covers $d_{n-1}$ in $D$ and, as $D$ is a lower set, also in $T$. Now, for any $d \in D$, if $d_m < d$ for all $m$ then $\downarrow d$ would be infinite; hence, we must have $d \leq d_n$ for some $n$ and then $d \in \downarrow d_n = \{d_k : k < n\}$.

Now, we shall see that the Noetherian Lift Lemma is related to a famous graph-theoretical tool, namely...
KL König’s Generalized Infinity Lemma:
If \( R \) is a locally finite relation and \( x \) is an element with \( xR^n \neq \emptyset \) for all \( n \), then there is a sequence \( (x_n) \) with \( x_0 = x \) and \( x_n R x_{n+1} \) for all \( n \).

In fact, we have:

**Theorem 5** The following postulates are mutually equivalent consequences of the Noetherian Lift Lemma (hence of the Ultrafilter Principle):

- **C\(_\omega \)<\(_\omega \)** The Axiom of Choice for Countable Families of Finite Sets.
- **U\(_\omega \)<\(_\omega \)** Countable unions of finite sets are countable.
- **DC\(_f \)** The Principle of Dependent Choices for Locally Finite Relations.
- **KL** König’s Infinity Lemma for Locally Finite Relations.
- **KT** König’s Infinity Lemma for \( \omega \)-Trees (of Words).

**Proof.** The equivalence of these statements can be found, more or less explicitly, in the literature on weak choice axioms (see, for example, Felscher [12, pp.121–122]). The following implication circles are readily verified:

\[
\begin{array}{c}
\text{DC}_f \\
\text{C}_{\omega}^{\omega} \quad \text{U}_{\omega}^{\omega} \\
\text{KL} \\
\text{KT}
\end{array}
\]

For \( U_{\omega} < \omega \Rightarrow KL \) consider the set \( Y \) of all \( x \) such that \( xR^n \neq \emptyset \) for all \( n \); by induction, these sets are finite, so their union \( U_x = \bigcup \{ xR^n : n < \omega \} \) is countable by \( U_{\omega} < \omega \). Furthermore, by local finiteness, the sets \( Y \cap xR \) are nonempty, and we may define a map \( \varphi : Y \rightarrow Y \) by taking for \( \varphi(x) \) the first element of \( Y \cap xR \) with respect to the well-order induced by any enumeration of \( U_x \). Then \( x_n = \varphi^n(x) \) has the desired properties.

What is of interest for our present study is that \( NL \) implies \( KT \). Let \( T \) be an \( \omega \)-tree and consider \( x \in T \) with \( F_n = xR^n \neq \emptyset \) for all \( n \), where \( R \) denotes the covering relation. Passing to the subtree \( \uparrow x \), we may assume that \( x \) is the least element of \( T \). By induction, the sets \( F_n \) are finite and disjoint; thus, \( (\uparrow F_n : n < \omega) \) is an ascending sequence in \( F \times T \) with no greatest element. Indeed, we have \( F_{n+1} = (xR^n)R \subseteq \uparrow F_n \), and assuming \( z \in F_n \subseteq \uparrow F_{n+1} \), we find a \( y \in xR^{n+1} \) with \( y \leq z \). But, since \( T \) is concatenated, \( y \leq z \) would force \( z \) to lie in some \( F_m \) with \( m > n \). Now, \( NL \) yields an ideal \( D \) of \( T \), hence an \( \omega \)-chain, with no greatest element – in other words, a properly ascending sequence \( x = x_0 R x_1 R x_2 \ldots \).

Finally, using the known fact that there are models of \( ZF \) set theory in which \( AC \) holds for all families of sets with fixed finite cardinality but \( KT \) fails (see e.g. Jech [19, Ch.7.4]), we conclude that

*all principles discussed in this paper are unprovable in \( ZF \).*
References


