SL$_2$-TILINGs AND TRIANGULATIONS OF THE STRIP

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Abstract. SL$_2$-tilings were introduced by Assem, Reutenauer, and Smith in connection with frieses and their applications to cluster algebras.

An SL$_2$-tiling is a bi-infinite matrix of positive integers such that each adjacent $2 \times 2$–submatrix has determinant 1.

We construct a large class of new SL$_2$-tilings which contains the previously known ones. More precisely, we show that there is a bijection between our class of SL$_2$-tilings and certain combinatorial objects, namely triangulations of the strip.

1. Introduction

Our main result is sufficiently simple that we can begin with its statement.

Main Theorem. There is a bijection between SL$_2$-tilings with enough ones and triangulations of the strip.

A triangulation of the strip is a structure of the kind shown in Figure 1. An SL$_2$-tiling is a bi-infinite matrix of positive integers such that each adjacent $2 \times 2$–submatrix has determinant 1, see Figure 2. An SL$_2$-tiling is said to have enough ones if each quadrant

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{triangulation.png}
\caption{A triangulation of the strip}
\end{figure}

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Figure 2. An SL$_2$-tiling with enough ones

\((< i, > j)\) and each quadrant \((> i, < j)\) contains the value 1. We use the notation

\[
(< i, > j) = \{ (x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x < i, y > j \}
\]

and similarly for other inequality signs.

**Remark 1.1.** We follow matrix convention when writing tilings so the \(x\)-coordinate increases from top to bottom and the \(y\)-coordinate increases from left to right.

The triangulation in Figure 1 corresponds to the SL$_2$-tiling in Figure 2 under the bijection of the Main Theorem.

Defining a map \(\Phi\) from triangulations to tilings is in fact easy by using the theory of Conway–Coxeter friezes as introduced in [5] and [6], but it is harder to show that it is a bijection.

Triangulations of the strip can be viewed as infinite simplicial complexes and are, in that sense, classic. We first saw them mentioned explicitly by Igusa and Todorov in [9, sec. 4.3].

SL$_2$-tilings were introduced by Assem, Reutenauer, and Smith in [1, sec. 1]. They were explored by Bergeron and Reutenauer in [2] and [12] and are closely related to cluster algebras, cluster categories, friezes, and quiver mutations. There is also a link to the T-systems of theoretical physics, see [7, sec. 2.2].

The idea to link SL$_2$-tilings and triangulations of the strip came when we studied the cluster category introduced in [9, exam. 4.1.4(3)]. We explain this briefly in Appendix A.

Note that the best previous result on existence of SL$_2$-tilings is the following, see [1, thm. 3].
**Theorem** (Assem, Reutenauer, and Smith). *An infinite zig-zag path of ones in the plane can be extended to an SL$_2$-tiling.*

This is a special case of our Main Theorem, see Remark 4.4.

As shown by Bergeron and Reutenauer in [2, sec. 1], an SL$_2$-tiling is *tame* in the sense that it has rank 2 when viewed as a matrix. It was observed to us by Christophe Reutenauer that this takes a pleasantly concrete form in the present situation. Let $\mathcal{T}$ be a triangulation of the strip with associated SL$_2$-tiling $t = \Phi(\mathcal{T})$. Let $C_j$ be the $j$th column of $t$. Then $\gamma_j C_j = C_{j-1} + C_{j+1}$ where $\gamma_j$ is the number of ‘triangles’ in $\mathcal{T}$ which are incident with the $j$th vertex on the upper edge of the strip. There is an analogous formula which links rows of $t$ to vertices on the lower edge of the strip. Note that the vertices are numbered according to Figure 3. We return to this observation in Remark 9.3.

The paper is organised as follows: Section 2 gives rigorous definitions. Section 3 is a brief reminder on Conway–Coxeter friezes. Section 4 constructs the map $\Phi$ from triangulations of the strip to SL$_2$-tilings with enough ones. Sections 5–8 show a number of properties of SL$_2$-tilings with enough ones. Section 9 uses this to define a map $\Psi$ from SL$_2$-tilings with enough ones to triangulations of the strip and shows that it is an inverse to $\Phi$. Appendix A explains the link to a cluster category by Igusa and Todorov.

### 2. Definitions

**Definition 2.1.** The *vertices of the strip* are the elements of two disjoint copies of $\mathbb{Z}$ denoted $\mathbb{Z}^o = \{ \ldots, -1^o, 0^o, 1^o, \ldots \}$ and $\mathbb{Z}_o = \{ \ldots, -1_o, 0_o, 1_o, \ldots \}$.

A *connecting arc* is an element of $\mathbb{Z}^o \times \mathbb{Z}_o$, and an *internal arc* is an element $(p^o, q^o) \in \mathbb{Z}^o \times \mathbb{Z}^o$ or $(p_o, q_o) \in \mathbb{Z}_o \times \mathbb{Z}_o$ with $p \leq q - 2$. The word *arc* means connecting or internal arc.

We interpret the vertices and the arcs geometrically according to Figure 3 which shows the connecting arc $(2^o, 3_o)$ and the internal arcs $(1^o, 4^o)$ and $(0_o, 2_o)$. Note that the vertices along the upper and lower edges of the strip are numbered in opposite directions.
Remark 2.2. Interpreting the arcs geometrically gives an obvious notion of when two arcs cross. For instance, the arcs $(2°, 3°)$ and $(1°, 4°)$ cross, but the arcs $(2°, 3°)$ and $(0°, 2°)$ do not, see Figure 3.

Note that arcs which only meet at their end points do not cross; in particular, Figure 1 shows a set of pairwise non-crossing arcs.

Definition 2.3. A triangulation of the strip is a maximal collection $T$ of pairwise non-crossing arcs with the property that for each $(i, j) \in \mathbb{Z} \times \mathbb{Z}$ we have

\[(p^o, q_o) \in T \text{ for some } (p, q) \in (< i, > j) \text{ and } \]
\[(p^o, q_o) \in T \text{ for some } (p, q) \in (> i, < j). \]

See Figure 1.

Definition 2.4. An $SL_2$-tiling $t$ is a map

$$\mathbb{Z} \times \mathbb{Z} \ni (i, j) \mapsto t_{ij} \in \{1, 2, 3, \ldots\}$$

such that

$$\begin{vmatrix}
  t_{ij} & t_{i,j+1} \\
  t_{i+1,j} & t_{i+1,j+1}
\end{vmatrix} = 1$$

for $(i, j) \in \mathbb{Z} \times \mathbb{Z}$.

The tiling $t$ has enough ones if, for each $(i, j) \in \mathbb{Z} \times \mathbb{Z}$, we have

\[t_{pq} = 1 \text{ for some } (p, q) \in (< i, > j) \text{ and } \]
\[t_{pq} = 1 \text{ for some } (p, q) \in (> i, < j). \]

See Figure 2. We will occasionally write $t(i, j)$ in place of $t_{ij}$ to avoid nested subscripts.

Remark 2.5. The similarity between equations (2) and (4) is no coincidence: If $T$ and $t$ correspond under the bijection of our main theorem, then $(i^o, j_o) \in T$ if and only if $t_{ij} = 1$.

See Proposition 4.3.

Example 2.6. Not every $SL_2$-tiling has enough ones as shown by the tiling in Figure 4.

It is defined by continuing the pattern in the shaded cells in the obvious way, then filling in the rest of the cells using Equation (3). One shows directly that all resulting values are positive integers and that the tiling has only a single occurrence of 1.

3. Reminder on Conway–Coxeter friese

Conway–Coxeter friese were introduced by the eponymous authors. We refer to them henceforth as friese. The main source is the companion papers [5] and [6], which are not always easy to cite as they provide only outline details. We sometimes cite [3] instead.
Definition 3.1. A partial \( \text{SL}_2 \)-tiling defined on a subset \( D \subseteq \mathbb{Z} \times \mathbb{Z} \) is a map
\[
D \ni (i, j) \mapsto t_{ij} \in \{1, 2, 3, \ldots\}
\]
satisfying Equation (3) when it makes sense.

In particular, let \( D \) be a diagonal band bounded by two parallel lines running ‘northwest’ to ‘southeast’. A friese is a partial \( \text{SL}_2 \)-tiling \( t \) defined on \( D \) such that \( t_{ij} = 1 \) for each \((i, j)\) on the edges of \( D \), see Figure 5. Note that the vertical width of \( D \) can be any positive integer and \( D \) can be placed anywhere in the plane.

Remark 3.2. Let \( P \) be an \((n+1)\)-gon with vertices 0, 1, \ldots, \( n \) and let \( D \) be a fixed diagonal band of vertical width \( n \). Then there is a bijective correspondence between triangulations of \( P \) and frieses on \( D \).

The correspondence is realised as follows: Fix a vertical line segment \( V \) from edge to edge of \( D \). Each triangulation \( X \) of \( P \) gives rise to a map from diagonals of \( P \) to positive integers as explained in [3, p. 172]. In [3] the value of the map on the diagonal from \( A \) to \( B \) is denoted by \((A, B)\) but we denote it by
\[
X(A, B)
\]
to emphasise its dependence on \( X \). The friese corresponding to \( X \) is defined by having the following values on \( V \), see Figure 5.
\[
X(1, 0), \ldots, X(n, 0).
\]
This determines the whole friese, see (10) in [5] and [6].
FIGURE 5. A friese. It is defined on a diagonal band $D$ and each entry on the edges of $D$ is 1. In particular $\mathcal{X}(1,0) = \mathcal{X}(n,0) = 1$

**Definition 3.3.** A fundamental region of a friese is the restriction of the friese to a triangle $F$ of the form shown in Figure 6. Note that the fundamental region includes a diagonal of 1’s along the base of $F$ and a 1 at its apex.

**Remark 3.4.** Strictly speaking, $D$ and $F$ consist of the grid points in a band and a triangle, but we will be lax about this to avoid verbosity.
Remark 3.5. Consider a friese defined on a diagonal band $D$. Let $\ell$ be a descending diagonal line down the middle of $D$. A fundamental region of the tiling can be reflected in $\ell$, and the fundamental region and its reflection can be translated along $\ell$. It was shown in [6, (21)] that the friese is covered by such translations as shown in Figure 7.

4. FROM TRIANGULATIONS OF THE STRIP TO $SL_2$-TILINGS

Construction 4.1. Let $\Sigma$ be a triangulation of the strip. We construct an $SL_2$-tiling with enough ones,

$$t = \Phi(\Sigma),$$

as follows.
Let \((i, j) \in \mathbb{Z} \times \mathbb{Z}\) be given and consider the arc \((i^\circ, j^\circ)\). Choose arcs \((p^\circ, q^\circ), (r^\circ, s^\circ) \in \Xi\) with \(p < i < r, s < j < q\); this is possible according to Equation (2) in Definition 2.3. Figure 8 shows the resulting situation.

Now \(\{p^\circ, \ldots, r^\circ, s^\circ, \ldots, q^\circ\}\) can be viewed as the vertices of a finite polygon \(P\) and the arcs situated between \((r^\circ, s^\circ)\) and \((p^\circ, q^\circ)\) can be viewed as the diagonals of \(P\). In particular, the arcs of \(\Xi\) situated between \((r^\circ, s^\circ)\) and \((p^\circ, q^\circ)\) form a triangulation \(\Xi_P\) of \(P\).

Define \(t\) in terms of the map from Remark 3.2 by setting
\[
t_{ij} = \Xi_P(i^\circ, j^\circ). \tag{5}
\]

**Remark 4.2.** In Construction 4.1 we started with \((i, j) \in \mathbb{Z} \times \mathbb{Z}\) and considered the arc \((i^\circ, j^\circ)\). This is an arbitrary choice and we could as well have considered \((j^\circ, i^\circ)\).
Proposition 4.3. Let $\mathcal{T}$ be a triangulation of the strip. Construction 4.1 gives a well-defined $\text{SL}_2$-tiling $t = \Phi(\mathcal{T})$ with enough ones. It has the property

$$t_{ij} = 1 \iff (i^o, j^o) \in \mathcal{T}.$$  

(6)

Proof. $t$ is well-defined: The definition of $t_{ij}$ involves the choice of two arcs $(p^o, q^o), (r^o, s^o) \in \mathcal{T}$. Another choice, say $(p'^o, q'^o), (r'^o, s'_o) \in \mathcal{T}$, gives another finite polygon $Q$ with vertices \{ $p'^o, \ldots, r'^o, s'_o, \ldots, q'^o$ \}, and the arcs in $\mathcal{T}$ which are situated $(r'^o, s'_o)$ and $(p'^o, q'^o)$ form a triangulation $\mathcal{T}_Q$ of $Q$. We must show $t_P(i^o, j^o) = t_Q(i^o, j^o)$. 

(7)

However, in the case shown in Figure 9 the triangulation $\mathcal{T}_Q$ can be obtained from $\mathcal{T}_P$ by gluing triangulated polygons to the edges $(r^o, s^o)$ and $(p^o, q^o)$ of $P$, so (7) follows from [3, lemmas 1 and 2(a)] by induction. The other possible cases have $(p^o, q^o)$ and $(p'^o, q'^o)$ interchanged and/or $(r^o, s^o)$ and $(r'^o, s'_o)$ interchanged; they are handled by the same means.

$t$ is an $\text{SL}_2$-tiling: The values of $t_P(-, -)$ are positive integers by Remark 3.2, so we just have to prove Equation (3) for $i, j \in \mathbb{Z}$. Definition 2.3 permits us to choose $(p^o, q^o), (r^o, s^o) \in \mathcal{T}$ with $p < i < i + 1 < r$, $s < j < j + 1 < q$. Then Equation (3) amounts to

$$| \begin{array}{cc} \mathcal{T}_P(i^o, j^o) & \mathcal{T}_P(i^o, (j + 1)^o) \\
\mathcal{T}_P((i + 1)^o, j^o) & \mathcal{T}_P((i + 1)^o, (j + 1)^o) \end{array} | = 1,$$

which is true since $\mathcal{T}_P(-, -)$ defines a frieze, see Remark 3.2.

The biimplication (6): Follows from $\mathcal{T}_P(i^o, j^o) = 1 \iff (i^o, j^o) \in \mathcal{T}_P$, see [6, (32)].

$t$ has enough ones: Follows from the last part of Definition 2.3 and the biimplication (6).

$\square$

Remark 4.4. It was shown in [1, thm. 3] that an infinite zig-zag path of ones in the plane can be extended to an $\text{SL}_2$-tiling. This is a special case of Construction 4.1: If $\mathcal{T}$ is a triangulation of the strip which has only connecting arcs, then it is easy to see from
Equation (6) that \( t = \Phi(\Xi) \) has a zig-zag path of ones. It is also not hard to see that any zig-zag path can be obtained from such a \( \Xi \).

5. **Properties of SL\(_2\)-tilings I: Ptolemy formulæ**

The idea of this section is to think of an SL\(_2\)-tiling \( t \) as a map

\[
(i^o, j^o) \mapsto t_{ij}
\]

from the set of connecting arcs to the set of positive integers. We will define two other maps

\[
(i^o, j^o) \mapsto c_{ij}, \quad (i^o, j^o) \mapsto d_{ij}
\]

from internal arcs to positive integers. Between them, \( t, c, \) and \( d \) can be thought of as a map \( \chi \) from the set of all arcs to the set of positive integers. We will show several instances of the Ptolemy formula

\[
\chi(a)\chi(a') = \chi(e)\chi(e') + \chi(f)\chi(f')
\]

when \( a, a' \) are crossing arcs enclosed in a quadrangle of arcs \( e, f, e', f' \) as shown schematically in Figure 10.

**Definition 5.1.** Let \( t \) be an SL\(_2\)-tiling and let \( i < j \) be integers. Choose an integer \( a \) and set

\[
c_{ij} = \begin{vmatrix} t_{ia} & t_{i,a+1} \\ t_{ja} & t_{j,a+1} \end{vmatrix}, \quad d_{ij} = \begin{vmatrix} t_{ai} & t_{aj} \\ t_{a+1,i} & t_{a+1,j} \end{vmatrix}.
\]

**Remark 5.2.** The determinants in Definition 5.1 are independent of the choice of \( a \). This follows from [12, prop. 11.2] because [2, prop. 1] implies that \( t \) is *tame* in the terminology of [2, sec. 1]. Note that

\[
c_{i,i+1} = d_{i,i+1} = 1
\]

for \( i \in \mathbb{Z} \) since \( t \) is an SL\(_2\)-tiling.
Remark 5.3. Consider a $2 \times n$–matrix with $n \geq 4$ and look at columns number $i, j, k, \ell$ for $i < j < k < \ell$.

\[
\begin{bmatrix}
\cdots & u_{1i} & \cdots & u_{1j} & \cdots & u_{1k} & \cdots & u_{1\ell} & \cdots \\
\cdots & u_{2i} & \cdots & u_{2j} & \cdots & u_{2k} & \cdots & u_{2\ell} & \cdots
\end{bmatrix}
\]

It is classic, and elementary to show, that we have the following Ptolemy formula.

\[
\begin{vmatrix}
u_{1i} & v_{1k} \\
u_{2i} & v_{2k}\end{vmatrix} \begin{vmatrix}
v_{1j} & v_{1\ell} \\
v_{2j} & v_{2\ell}\end{vmatrix} = \begin{vmatrix}
v_{1i} & v_{1j} & v_{1k} & v_{1\ell} \\
v_{2i} & v_{2j} & v_{2k} & v_{2\ell}\end{vmatrix} + \begin{vmatrix}
v_{1i} & v_{1j} & v_{1k} & v_{1\ell} \\
v_{2i} & v_{2j} & v_{2k} & v_{2\ell}\end{vmatrix}.
\]

This implies the following result, which says that the Ptolemy formula holds for crossing internal arcs such as the ones in Figure 11.

Proposition 5.4. Let $t$ be an SL$_2$-tiling, $i < j < k < \ell$ integers. Then

\[
c_{ik}c_{j\ell} = c_{ij}c_{k\ell} + c_{i\ell}c_{jk}, \quad d_{ik}d_{j\ell} = d_{ij}d_{k\ell} + d_{i\ell}d_{jk}.
\]

The following proposition says that the Ptolemy formula holds for an internal arc crossing a connecting arc as in Figure 12. It can be proved from Definition 5.1 by direct computation.
Proposition 5.5. Let \( t \) be an \( SL_2 \)-tiling. If \( i < j < k \) and \( a \) are integers, then

\[
  t_{ja}c_{ik} = t_{ia}c_{jk} + t_{ka}c_{ij}, \quad t_{aj}d_{ik} = t_{ai}d_{jk} + t_{ak}d_{ij}.
\]

The following proposition shows an easy consequence.

Proposition 5.6. Let \( t \) be an \( SL_2 \)-tiling, \( i < j \) integers. Then \( c_{ij} \) and \( d_{ij} \) are positive integers.

\textbf{Proof.} It is clear from Definition 5.1 that \( c_{ij} \) and \( d_{ij} \) are integers, so it remains to see that \( c_{ij}, d_{ij} > 0 \). We have \( c_{i,i+1} = d_{i,i+1} = 1 > 0 \) by Remark 5.2. We proceed by induction on \( j - i \geq 1 \). Since \( i < j < j + 1 \), Proposition 5.5 gives

\[
  c_{i,j+1} = \frac{t_{ia}c_{j,j+1} + t_{j+1,a}c_{ij}}{t_{ja}} = \frac{t_{ia} + t_{j+1,a}c_{ij}}{t_{ja}}
\]

for each \( a \in \mathbb{Z} \). The induction implies that this expression is positive, and \( d_{ij} \) is handled analogously. \( \square \)

Finally, we show that the Ptolemy formula holds for crossing connecting arcs as in Figure 13. This formula can be written by means of a determinant.

Proposition 5.7. Let \( t \) be an \( SL_2 \)-tiling, \( i < j \) and \( p < q \) integers. Then

\[
\begin{vmatrix}
  t_{ip} & t_{iq} \\
  t_{jp} & t_{jq}
\end{vmatrix} = c_{ij}d_{pq}.
\]

In particular, it follows from Proposition 5.6 that the determinant on the left hand side is a positive integer.

\textbf{Proof.} If \( i, j \) are integers with \( i \leq j - 2 \), then \( i < j - 1 < j \) and Proposition 5.5 gives

\[
  t_{j-1,a}c_{ij} = t_{ia}c_{j-1,j} + t_{ja}c_{i,j-1}
\]

for each integer \( a \). Combining with Remark 5.2 gives

\[
  t_{ia} = t_{j-1,a}c_{ij} - t_{ja}c_{i,j-1}.
\]
This equation remains true for \( i = j - 1 \) if we make the temporary assignment \( c_{ii} = 0 \), so it holds for \( i < j \) and gives the second equality in the following computation.

\[
\begin{vmatrix}
t_{ip} & t_{iq} \\
t_{jp} & t_{jq}
\end{vmatrix} = t_{ip}t_{jq} - t_{iq}t_{jp} = (t_{j-1,p}c_{ij} - t_{jp}c_{i,j-1})t_{jq} - (t_{j-1,q}c_{ij} - t_{jq}c_{i,j-1})t_{jp} = (t_{j-1,p}t_{jq} - t_{j-1,q}t_{jp})c_{ij} = c_{ij}d_{pq}.
\]

\[\square\]

6. Properties of \( \text{SL}_2 \)-tilings II: Forbidden values

This section shows two consequences of the Ptolemy formulae of Section 5.

**Proposition 6.1.** Let \( t \) be an \( \text{SL}_2 \)-tiling, \( n \) a positive integer. If \( i \) is fixed then \( t_{ij} = n \) for at most finitely many values of \( j \). If \( j \) is fixed then \( t_{ij} = n \) for at most finitely many values of \( i \).

**Proof.** Fix \( i \) and suppose that there is an increasing sequence of integers

\[ j < k < \cdots \tag{8} \]

with \( t_{ij} = t_{ik} = \cdots = n \). The two first terms in the sequence give

\[ d_{jk} = \begin{vmatrix} t_{i-1,j} & t_{i-1,k} \\ t_{ij} & t_{ik} \end{vmatrix} = \begin{vmatrix} t_{i-1,j} & t_{i-1,k} \\ n & n \end{vmatrix} = n(t_{i-1,j} - t_{i-1,k}) \]

and since \( d_{jk} > 0 \) by Proposition 5.6, this implies \( t_{i-1,j} > t_{i-1,k} \). Using the subsequent terms in (8) gives a string of inequalities \( t_{i-1,j} > t_{i-1,k} > \cdots \). Since the values of \( t \) are positive integers, this implies that the sequence (8) is only finitely long.

The case of a decreasing sequence of integers \( j > k > \cdots \) is handled analogously using

\[ d_{kj} = \begin{vmatrix} t_{ik} & t_{ij} \\ t_{i+1,k} & t_{i+1,j} \end{vmatrix}. \]

This shows the first claim and the second one is shown using \( c_{jk} \). \[\square\]

**Proposition 6.2.** Let \( t \) be an \( \text{SL}_2 \)-tiling. If \( t_{ij} = 1 \) then \( t \) does not have the value 1 in the quadrants \( (< i, < j) \) and \( (> i, > j) \).

**Proof.** If \( t_{xy} = 1 \) for some \( (x, y) \in (< i, < j) \) then

\[
\begin{vmatrix} t_{xy} & t_{xj} \\
t_{iy} & t_{ij} \end{vmatrix} = \begin{vmatrix} 1 & t_{xj} \\
t_{iy} & 1 \end{vmatrix} = 1 - t_{xj}t_{iy}. \]
On the one hand, the determinant is positive by Proposition 5.7. On the other hand, \( t_{xj} \) and \( t_{iy} \) are positive integers so \( 1 - t_{xj} t_{iy} \leq 0 \), a contradiction. The case \((x, y) \in (i, j)\) is handled analogously.

\[ \square \]

7. Properties of \( SL_2 \)-tilings III: A link to Conway–Coxeter frieses

Let us remind the reader that Section 3 gave a few salient facts on frieses. This section shows a link between frieses and \( SL_2 \)-tilings.

The following lemma provides a converse to Remark 3.5. It is certainly well-known but we do not know a reference so give a proof.

**Lemma 7.1.** Let \( D \) be a diagonal band and let \( F \) be a triangle inside \( D \) as shown in Figure 6.

Let \( t \) be a partial \( SL_2 \)-tiling defined on \( F \) such that \( t_{ij} \) is equal to 1 along the base of \( F \) and at its apex, again as shown in Figure 6.

Then there is a friese defined on \( D \) which agrees with \( t \) on \( F \).

**Proof.** Let \( V \) denote the vertical line segment bounded by asterisks in Figure 6 and suppose that it contains \( n \) grid points. By [11, thm. 3.1] there is a map

\[ D \ni (i, j) \mapsto u_{ij} \in \mathbb{Q}(x_1, \ldots, x_n) \]

with the following properties.

(i) The values of \( u \) are Laurent polynomials.

(ii) We have

\[ \begin{vmatrix} u_{ij} & u_{i,j+1} \\ u_{i+1,j} & u_{i+1,j+1} \end{vmatrix} = 1 \]

when the determinant makes sense.

(iii) If \((i, j)\) is on one of the edges of \( D \) then \( u_{ij} = 1 \).

(iv) The values of \( u \) on \( V \) are \( x_1, \ldots, x_n \).

(v) Restricting \( u \) to \( F \) gives a region from which \( u \) can be recovered by the recipe of Figure 7.

Set the variables \( x_i \) equal to the values of \( t \) on \( V \). These values are non-zero integers, so by (i) this gives a map

\[ D \ni (i, j) \mapsto v_{ij} \in \mathbb{Q} \]

Each determinant

\[ \begin{vmatrix} v_{ij} & v_{i,j+1} \\ v_{i+1,j} & v_{i+1,j+1} \end{vmatrix} \]

which makes sense is 1 by (ii), and (iii) and (iv) imply that \( t \) and \( v \) are equal on \( V \). Hence \( t \) and \( v \) are equal on \( F \), see (10) in [5] and [6].
Figure 14. $t$ is an SL$_2$-tiling with $t_{jp} = t_{iq} = 1$. There is a friese agreeing with $t$ on the rectangle $R$ by Proposition 7.2

The values of $t$ on $F$ are positive integers, so the same holds for $v$. By (v), all the values of $v$ are hence positive integers, so $v$ is a friese with the property claimed in the lemma. \qed

**Proposition 7.2.** Let $t$ be an SL$_2$-tiling, $i \leq j$ and $p \leq q$ integers such that $(i, j) \neq (p, q)$ and $t_{jp} = t_{iq} = 1$.

Then there is a friese as shown in Figure 14 which agrees with $t$ on the rectangle $R = (i \ldots j, p \ldots q)$, where we use the notation

$$(i \ldots j, p \ldots q) = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid i \leq x \leq j, p \leq y \leq q\}.$$

**Proof.** By Lemma 7.1, all we need is to take the restriction of $t$ to $R$ and extend it to a partial SL$_2$-tiling, which is defined on a triangle $F$ as in Figure 6 and has value 1 on the base and at the apex of $F$. We do so explicitly in Figure 15. The extension is accomplished by filling in two smaller triangles adjacent to $R$ by using $c$ and $d$ from Definition 5.1.

The extension is a partial SL$_2$-tiling because it consists of positive integers by Proposition 5.6, and because all resulting adjacent $2 \times 2$-submatrices which make sense have determinant equal to 1. The last claim follows from Propositions 5.4 and 5.5. It is elementary but tedious to check this and we omit the details. \qed
This section draws on the results of the two previous sections to show that if $t$ is an $SL_2$-tiling with enough ones, then there is a zig-zag path in the plane which contains all the $(i, j)$ with $t_{ij} = 1$. The terminology is made precise in Proposition 8.2.

The following proposition uses the notation for quadrants exemplified by Equation (1) in the introduction, and a similar notation for half lines, for instance

$$(< j, p) = \{ (x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x < j, y = p \}.$$ 

**Proposition 8.1.** Let $t$ be an $SL_2$-tiling with $t_{jp} = 1$.

(i) If $t$ has the value 1 somewhere in the quadrant $(< j, > p)$, then it also has the value 1 somewhere on the half line $(< j, p)$ or somewhere on the half line $(j, > p)$, but not both.
(ii) If $t$ has the value 1 somewhere in the quadrant $(> j, < p)$, then it also has the value 1 somewhere on the half line $(> j, p)$ or somewhere on the half line $(j, < p)$, but not both.

Proof. We only prove (i) as (ii) has an analogous proof.

The last part of (i) (“not both”) is immediate from Proposition 6.2.

To show the first part of (i), assume that it fails. Then we have that $t_{jp} = 1$, there is $(i, q) \in (< j, > p)$ with $t_{iq} = 1$, and $t$ is different from 1 on $(< j, p)$ and on $(j, > p)$.

If we choose $(i, q) \in (< j, > p)$ as close as possible to $(j, p)$ then $t_{jp} = t_{iq} = 1$ are the only occurrences of 1 in the rectangle $R = (i \ldots j, p \ldots q)$. (9)

By Proposition 7.2 there is a friese $v$ which agrees with $t$ on $R$, as shown in Figure 14. The friese corresponds to a triangulation $\mathfrak{T}_P$ of a finite polygon $P$, see Remark 3.2. Consider the triangle $F$ such that the restriction of $v$ to $F$ is the fundamental region shown in Figure 16. There is a bijective correspondence between diagonals in $P$ and grid points in $F$, see [3, p. 172]. Note that in this context, the edges of $P$ are considered to be diagonals and they correspond to the grid points along the base and at the apex of $F$. Moreover, the diagonal corresponding to $(i, p)$ crosses precisely the diagonals corresponding to the grid points inside the box in Figure 16.

For $(x, y) \in F$ we have that $t_{xy} = 1$ if and only if the diagonal corresponding to $(x, y)$ is in $\mathfrak{T}_P$, see [6, (32)]. By Equation (9), none of the $t_{xy}$ in the box are 1, so none of the corresponding diagonals are in $\mathfrak{T}_P$. Hence the diagonal corresponding to $(i, p)$ must be in $\mathfrak{T}_P$ whence $t_{ip} = 1$, but this contradicts Equation (9). □

Proposition 8.2. Let $t$ be an $\text{SL}_2$-tiling with enough ones. There exist $(x_\alpha, y_\alpha) \in \mathbb{Z} \times \mathbb{Z}$ for $\alpha \in \mathbb{Z}$ with the following properties.

(i) $t_{xy} = 1 \iff (x, y) = (x_\alpha, y_\alpha)$ for some $\alpha$.

(ii) For each $\alpha$, either

(a) $x_{\alpha+1} < x_\alpha$ and $y_{\alpha+1} = y_\alpha$, or

(b) $x_{\alpha+1} = x_\alpha$ and $y_{\alpha+1} > y_\alpha$.

(iii) When $\alpha$ goes to $\infty$ or $-\infty$, there are infinitely many shifts between options (a) and (b).

The $(x_\alpha, y_\alpha)$ are unique with these properties, up to adding a constant integer to $\alpha$.

Proof. Uniqueness is straightforward so we show existence. Pick $(x_0, y_0)$ with $t(x_0, y_0) = 1$. Now suppose that $(x_\alpha, y_\alpha)$ have been defined for $|\alpha| \leq A$ such that $t(x_\alpha, y_\alpha) = 1$ for each $\alpha$. 
To define \((x_{A+1}, y_{A+1})\), note that \(t(x_A, y_A) = 1\) and that \(t\) has the value 1 somewhere in the quadrant \((< x_A, y_A)\) by Definition 2.4. Proposition 8.1(1) says that either

1. \(t\) has the value 1 somewhere on the half line \((< x_A, \cdot)\) or
2. \(t\) has the value 1 somewhere on the half line \((\cdot, y_A)\),

but not both.

If (1) occurs then let \(x_{A+1}\) be maximal such that \(x_{A+1} < x_A\) and \(t(x_{A+1}, y_A) = 1\). Set \(y_{A+1} = y_A\).

If (2) occurs then let \(y_{A+1}\) be minimal such that \(y_{A+1} > y_A\) and \(t(x_A, y_{A+1}) = 1\). Set \(x_{A+1} = x_A\).

To define \((x_{-A-1}, y_{-A-1})\), use an analogous method based on Proposition 8.1(2).

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**Figure 16. A fundamental region**

To define \((x_{A+1}, y_{A+1})\), note that \(t(x_A, y_A) = 1\) and that \(t\) has the value 1 somewhere in the quadrant \((< x_A, y_A)\) by Definition 2.4. Proposition 8.1(1) says that either

1. \(t\) has the value 1 somewhere on the half line \((< x_A, \cdot)\) or
2. \(t\) has the value 1 somewhere on the half line \((\cdot, y_A)\),

but not both.

If (1) occurs then let \(x_{A+1}\) be maximal such that \(x_{A+1} < x_A\) and \(t(x_{A+1}, y_A) = 1\). Set \(y_{A+1} = y_A\).

If (2) occurs then let \(y_{A+1}\) be minimal such that \(y_{A+1} > y_A\) and \(t(x_A, y_{A+1}) = 1\). Set \(x_{A+1} = x_A\).

To define \((x_{-A-1}, y_{-A-1})\), use an analogous method based on Proposition 8.1(2).
Now consider properties (i)–(iii) in the lemma. The definition of the \((x_\alpha, y_\alpha)\) makes it clear that they satisfy (ii) and \(\Leftarrow\) in (i). Property (iii) also holds, for if it failed then \(t\) would contradict Proposition 6.1.

To see \(\Rightarrow\) in property (i), note that by (iii), the \((x_\alpha, y_\alpha)\) define an infinite zig-zag path in the plane, see Figure 17. At each corner of the zig-zag path, \(t\) has the value 1. Each such corner prevents \(t\) from having the value 1 in two whole quadrants by Proposition 6.2; these quadrants are also shown in Figure 17. Between them, the quadrants cover the whole plane except for the zig-zag path, so \(t_{xy} = 1\) implies that \((x, y)\) is on the zig-zag path. In fact, we
must even have \((x, y) = (x_\alpha, y_\alpha)\) for some \(\alpha\), as claimed. Namely, the construction shows that going from \((x_A, y_A)\) to \((x_{A+1}, y_{A+1})\) is the same as going to the “next” place on the zig-zag path where \(t\) has the value 1. So the only \((x, y)\) on the zig-zag path with \(t_{xy} = 1\) are the \((x_\alpha, y_\alpha)\).

\[\square\]

9. From \(SL_2\)-tilings to triangulations of the strip

**Construction 9.1.** Let \(t\) be an \(SL_2\)-tiling with enough ones. We construct a triangulation of the strip,

\[\mathfrak{T} = \Psi(t),\]

as follows.

Consider the \((x_\alpha, y_\alpha)\) for \(\alpha \in \mathbb{Z}\) from Proposition 8.2, and start by including the connecting arcs \(((x_\alpha)^o, (y_\alpha)_o)\) in \(\mathfrak{T}\). They are pairwise non-crossing by Proposition 8.2(ii), and Proposition 8.2(iii) implies that \(\mathfrak{T}\) will satisfy the last part of Definition 2.3.

We complete the construction of \(\mathfrak{T}\) by including further arcs for each value of \(\alpha \in \mathbb{Z}\) as follows:

Consider an \(\alpha\). Suppose that \(x_{\alpha+1} < x_\alpha\) and \(y_{\alpha+1} = y_\alpha\) as in Proposition 8.2(ii)(a); the alternative case \(x_{\alpha+1} = x_\alpha\) and \(y_{\alpha+1} > y_\alpha\) is handled analogously. Figure 18 shows part of \(\mathfrak{T}\) as constructed so far.

![Figure 18. A part of \(\mathfrak{T}\) as constructed so far](image)

This part of \(\mathfrak{T}\) can be viewed as a finite polygon \(P\) with vertices

\[(x_{\alpha+1})^o, (x_{\alpha+1} + 1)^o, \ldots, (x_\alpha - 1)^o, (x_\alpha)^o, (y_\alpha)_o\]

and we will construct a triangulation \(\mathfrak{T}_P\) of \(P\) whose diagonals can be viewed as arcs which will be included in \(\mathfrak{T}\).
To construct $\mathfrak{F}_P$, note $t(x_a, y_a) = 1$ and $t(x_{a+1}, y_a) = t(x_{a+1}, y_{a+1}) = 1$. So Proposition 7.2 says that there is a friese $v$ on a diagonal band $D$ such that on the line segment $(x_{a+1} \ldots x_a, y_a)$ extends to a friese $v$ on the diagonal band $\{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x_{a+1} \leq x \leq x_a, \ y = y_a\}$. 

This is shown in Figure 19. There is a bijective correspondence between frieses and triangulations as explained in Remark 3.2. In the case at hand, the bijection says that a
triangulation $\mathfrak{T}_P$ of $P$ corresponds to a friese on $D$ which has the following values on the line segment $(x_{\alpha+1}, x_\alpha, y_\alpha)$.

$\mathfrak{T}_P((x_{\alpha+1})^\circ, (y_\alpha)_\circ), \mathfrak{T}_P((x_{\alpha+1}+1)^\circ, (y_\alpha)_\circ), \ldots,$

$\mathfrak{T}_P((x_\alpha-1)^\circ, (y_\alpha)_\circ), \mathfrak{T}_P((x_\alpha)^\circ, (y_\alpha)_\circ)$

Define $\mathfrak{T}_P$ by requiring that its friese agrees with $v$, that is,

$$\mathfrak{T}_P(x^\circ, (y_\alpha)_\circ) = t(x, y_\alpha) \text{ for } x_{\alpha+1} \leq x \leq x_\alpha.$$

Note that carrying out this construction for each $\alpha \in \mathbb{Z}$ does indeed turn $\mathfrak{T}$ into a maximal set of pairwise non-crossing arcs. Namely, the connecting arcs $((x_\alpha)^\circ, (y_\alpha)_\circ)$ divide the strip into doubly infinite sequence of finite polygons like $P$, and the construction includes a triangulation of each of these into $\mathfrak{T}$.

The following is a more concise version of the Main Theorem from the introduction.

**Theorem 9.2.** The maps $\Phi$ and $\Psi$ from Constructions 4.1 and 9.1 are inverse bijections between the $\text{SL}_2$-tilings with enough ones and the triangulations of the strip.

**Proof.** The proofs that $\Psi \circ \Phi$ and $\Phi \circ \Psi$ are the identity are closely related, so we only show the proof for $\Psi \circ \Phi$.

Let $\mathfrak{T}$ be a triangulation of the strip and write $t = \Phi(\mathfrak{T})$ and $\mathfrak{U} = \Psi(t)$. We must show $\mathfrak{U} = \mathfrak{T}$.

Let $((x_\alpha)^\circ, (y_\alpha)_\circ)$ be the connecting arcs in $\mathfrak{T}$. Equation (6) in Proposition 4.3 says that $t_{xy} = 1$ if and only if $(x, y) = (x_\alpha, y_\alpha)$ for some $\alpha$. Hence by Proposition 8.2 we can assume that the $(x_\alpha, y_\alpha)$ are defined for $\alpha \in \mathbb{Z}$ and have the properties listed in Proposition 8.2.

Construction 9.1 implies that the connecting arcs $((x_\alpha)^\circ, (y_\alpha)_\circ)$ are also present in $\mathfrak{U}$. Now consider the connecting arcs with indices $\alpha$ and $\alpha + 1$. Suppose that $x_{\alpha+1} < x_\alpha$ and $y_{\alpha+1} = y_\alpha$ as in Proposition 8.2(ii)(a); the alternative case $x_{\alpha+1} = x_\alpha$ and $y_{\alpha+1} > y_\alpha$ is handled analogously. The connecting arcs in question are shown in Figure 18, and the part of the strip between them can be considered as a finite polygon $P$ with the vertices listed in Equation (10).

The arcs of $\mathfrak{T}$ which are inside $P$ give a triangulation $\mathfrak{T}_P$ of $P$. Similarly, $\mathfrak{U}$ gives a triangulation $\mathfrak{U}_P$ of $P$. Since the connecting arcs divide the strip into finite polygons like $P$, it is enough to show $\mathfrak{T}_P = \mathfrak{U}_P$ to finish the proof.

Equation (5) in Construction 4.1 implies

$$t(x, y_\alpha) = \mathfrak{T}_P(x^\circ, (y_\alpha)_\circ) \text{ for } x_{\alpha+1} \leq x \leq x_\alpha.$$

On the other hand, Equation (11) in Construction 9.1 says

$$\mathfrak{U}_P(x^\circ, (y_\alpha)_\circ) = t(x, y_\alpha) \text{ for } x_{\alpha+1} \leq x \leq x_\alpha.$$
So

$$\mathcal{T}_P(x^a, (y_a)_o) = \mathcal{U}_P(x^a, (y_a)_o) \quad \text{for} \quad x_{\alpha+1} \leq x \leq x_\alpha.$$  

That is, the frieses corresponding to $\mathcal{T}_P$ and $\mathcal{U}_P$ agree on the line segment $(x_{\alpha+1} \ldots x_\alpha, y_a)$. But this segment reaches from one edge of the fries to the other, so the fries are the same by (10) in [5] and [6]. Hence $\mathcal{T}_P = \mathcal{U}_P$ by (28) and (29) in [5] and [6] as desired. $\square$

**Remark 9.3.** We can now explain the observation by Christophe Reutenauer reproduced in the introduction. Let $\mathcal{T}$ be a triangulation of the strip and set $t = \Phi(\mathcal{T})$. Remark 5.2 and Proposition 5.5 give $t_{ja}c_{j-1,j+1} = t_{j-1,a} + t_{j+1,a}$, that is,

$$c_{j-1,j+1}C_j = C_{j-1} + C_{j+1}$$

where $C_j$ denotes the $j$th column of $t$. One can show that

$$c_{j-1,j+1} = \mathcal{T}_P((j - 1)^o, (j + 1)^o)$$

when $P$ is a suitable finite polygon containing $((j - 1)^o, (j + 1)^o)$ as a diagonal, cf. Construction 4.1. However, the right hand side of this equation is precisely the number of triangles in $\mathcal{T}_P$ incident with the vertex $j^o$, see [5, Introduction]. Hence it is the number of ‘triangles’ of $\mathcal{T}$ incident with $j^o$.

**Appendix A. A link to a cluster category of Igusa and Todorov**

Igusa and Todorov introduced a certain cluster category in [9, exam. 4.1.4(3)], see also [9, sec. 4.3]. It will be denoted by $\mathcal{C}$ and it categorifies the strip: Its set of isomorphism classes of indecomposable objects, $\text{ind} \mathcal{C}$, is in bijection with the set of arcs and there are non-trivial extensions between indecomposable objects $a$ and $b$ if and only if their arcs cross.

It is shown in [8] that if $\mathcal{T}$ is a triangulation of the strip, then the arcs of $\mathcal{T}$ correspond to a set of indecomposable objects of $\mathcal{C}$ whose additive closure is a cluster tilting subcategory $\mathbb{T}$. By the Caldero–Chapoton formula, such a $\mathbb{T}$ gives a cluster map

$$\rho : \text{obj} \mathcal{C} \to \mathbb{Q}(x_t)_{t \in \text{ind} \mathbb{T}}.$$  

See [4, sec. 4] for the definition of cluster maps and [10] for details of how the Caldero–Chapoton formula works for cluster tilting subcategories with infinitely many isomorphism classes of indecomposable objects.

The methods of [10, sec. 6] can be adapted to show that $\rho(t) = x_t$ for $t \in \text{ind} \mathbb{T}$, that the values of $\rho$ are Laurent polynomials, and that each of these has positive integer coefficients in the numerator. Hence, setting each $x_t$ equal to 1 turns $\rho$ into a map

$$\chi : \text{obj} \mathcal{C} \to \{ 1, 2, 3, \ldots \}.$$
One can obtain an $SL_2$-tiling from $\chi$. Namely, given $(i, j) \in \mathbb{Z} \times \mathbb{Z}$, the arc $(i^\circ, j^\circ)$ can be viewed as an indecomposable object of $\mathcal{C}$ and we set
\[ t_{ij} = \chi(i^\circ, j^\circ). \]

To see that this is an SL$_2$-tiling, note that it follows from [9, lem. 4.2.1] that in the triangulated category $\mathcal{C}$ we have
\[ \dim \operatorname{Ext} \left( ((i + 1)^\circ, (j + 1)^\circ), (i^\circ, j^\circ) \right) = \dim \operatorname{Ext} \left( (i^\circ, j^\circ), ((i + 1)^\circ, (j + 1)^\circ) \right) = 1. \]

Moreover, [9, prop. 4.2.12] gives the Auslander–Reiten triangle
\[ (i^\circ, j^\circ) \to ((i + 1)^\circ, j^\circ) \oplus (i^\circ, (j + 1)^\circ) \to ((i + 1)^\circ, (j + 1)^\circ) \]
while there is a non-split distinguished triangle
\[ ((i + 1)^\circ, (j + 1)^\circ) \to 0 \to (i^\circ, j^\circ) \]
by the formula $\Sigma((i + 1)^\circ, (j + 1)^\circ) = (i^\circ, j^\circ)$, see [9, sec. 4.2.1]. Note that the connecting map of the second triangle is the identity morphism of $(i^\circ, j^\circ)$.

By properties (M2) and (M3) of cluster maps stated in [4, sec. 4], the last three displayed formulae imply
\[ \chi((i + 1)^\circ, (j + 1)^\circ)\chi(i^\circ, j^\circ) = \chi\left( ((i + 1)^\circ, j^\circ) \oplus (i^\circ, (j + 1)^\circ) \right) + \chi(0) \]
\[ = \chi((i + 1)^\circ, j^\circ)\chi(i^\circ, (j + 1)^\circ) + 1 \]
whence $t$ satisfies
\[ t_{i+1,j+1}t_{ij} = t_{i+1,j}t_{i,j+1} + 1 \]
so $t$ is an SL$_2$-tiling.

Further use of the methods of [10, sec. 6] shows that $t$ is in fact $\Phi(\Sigma)$ from Construction 4.1. However, it is not clear if the categorical methods can be used to show that $\Phi$ is injective or surjective.

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References


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